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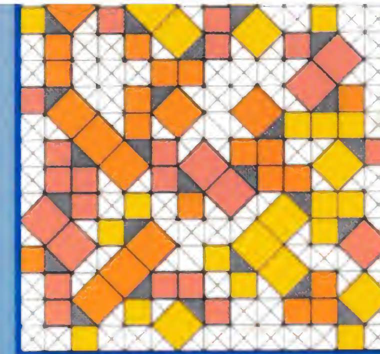

INNOVATIVE STRATEGIES OF TEACHING SCHOOL MATHEMATICS

PRIMARY SCHOOL STAGE

AN AMTI PUBLICATION

SPONSORED BY

Prof. T. R. Raghavasastri Memorial Trust

EMPOWERMENT OF PUPIL - TEACHER RELATIONSHIP

&

PARENT - CHILD RELATIONSHIP IN MATH LEARNING

The Association of
Mathematics Teachers of India

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BETWEEN US

Dear reader,

As announced earlier, we place in your hands the first volume of innovative strategies that may be adopted, adapted at the primary level. The material is an improved version of our practical experiences with children and teachers. More details of how this has been evolved may be found in the introductory chapter that follows.

We are grateful to *Prof. T.R. Raghava Sastri Memorial Trust* for deciding to fund the publication by allocating Rs. 1,00,000/- and for depositing Rs. 25000/- already.

Sri P.K.Srinivasan, the legendary Mathematics Education Exponent, a founder member of our association has accepted this onerous responsibility of editing these volumes. Even a cursory glance at this book will convince the reader the thought and expository techniques brought to surface by this fertile young mind at this ripe old age with total dedication to motivate the aspirants to take similar stimulated interest to enliven the class room relieving the children of the drudgery as it is practised today.

For *Sri G. Narayanan* and *Ku. R. Aruna* special thanks are offered for getting the type set in record time including diagrams of novel nature. We thank *M/S M.K. Graphics* and *J.V. Printers* for executing the printing process within minimum possible time.

We welcome feed back for serving the cause of excellence in Mathematics Education in our society.

With kind regards / best wishes,

Yours sincerely,

(M. MAHADEVAN)

Secretary,
-The A MTI

BACK DROP
CURTAIN RAISER

In the early nineties, *Mr. P.K. Srinivasan* was invited by *Mr. Anil Bordia,* the then Education Secretary, Ministry of Human Resources Development, Government of India, New Delhi to meet him in response to his letter on innovative strategy of instruction of school algebra from primary itself.

In the interview, the Education Secretary observed that excellence in school education cannot be secured without innovation in teaching practices relating to mathematics. He sanctioned two projects:

(i) Pattern and Design Language Approach in teaching school algebra from primary itself (alloted to the Department of Education, University of Madras, headed then by *Prof. P.S. Balasubramanian*) and

(ii) Improvement of mathematics Education in schools through four-day residential workshops. (alloted to the Association of mathematics Teachers of India with Prof. G. Rangan of Ramanujan Institute of Advance study in mathematics, Chennai – 5, as coordinator.) Mr. P.K. Srinivasan was associated with all these projects as chief Resource person.

(iii) While the first MHRD funded two year project (92-94) was confined to schools drawn from different economic strata of society in the city of Chennai, the second one involving two project meets each at primary, middle, high and

higher secondary level was held all over the country. All the projects were documented and sent to the Ministry.Created by

(iv) In the first project, *Mr. P.K. Srinivasan* acted as chief coordinator while *Dr.P.S. Balasubramanian* was the Director and in the second project covering eight workshops, *Dr. G. Rangan* was the chief coordinator with *Mr. P.K. srinivasan* as the chief Resource Person.

(v) It is intended to bring out five publications of Guidelines emerging from two interaction workshops on Math learning at
   (i) Primary school level
   (ii) Middle school level
   (iii) High school level
   (iv) Higher Secondary school level and
   (v) School Algebra experiences of responding schools through Pattern and Design Language from class III (updated later as Mathelang from preschool years).

Two workshops held at each of the four levels were so run that the first workshop saw introduction of innovative strategies in one part of the country and in the second workshop affirmation of the strategies adopted with refinements made where necessary, in another part of the country. At each workshop, there were resource persons from the area selected, teachers and pupils whose reactions were secured in writing besides project reports of teacher participants on non-routine themes.

Primary mathematics workshops:

The first one was held at *Ranipet* 5th to 8th March 1992 and the second one at *Kancheepuram* 9th to 12th January 1995.

Dissemination:

Curricular changes have taken place since; but they scarcely reflect the successfully demonstrated innovations in real class room situation involving teachers and pupils who have welcomed the innovations that improve the image of mathematics, arising out of authentic learning as opposed to authoritarian learning. When the child develops a sense of ownership over what he or she has come to think and do, the learning is authentic. When learning is authoritarian, the child has acquired knowledge not out of conviction but out of conformity to what is dictated.

Through authentic learning a child discovers that there is more in him or her, and s/he looks forward passionately to facilities and guidelines to see his or her way.

The subjects in school curriculum can be classified as looker-subjects, listener-subjects, doer-subjects, looker and listener subject, looker, listener and doer subjects. What are looker subjects? Writing and Art. What are listener subjects? Music and Language. What are looker and listener subjects? Dance and Drama. What are looker listener and doer subjects? Science and Mathematics. Since mathematics is required to be learnt more by looking and listening in present day schooling than by doing, learning is authoritarian.

The learner when challenged can be seen apologetic.

Since mathematics is man-made and need based, mathematical core of the child does not get nurtured with no climate for hands on experiences. As it is today, learning is by rule and drill at the behest of the teacher. It is unfortunate that the teacher himself or herself is a product of this system involving rule and drill. If at all a practising teacher has retrained herself/himself to help formation of concepts that gets emerged through involvement in situations reflecting multi embodiment principle and multipurpose use of materials s/he has to function in vacuum in the absence of encouragement from school authorities, public or private, and of understanding and collaboration from colleagues, the parents by and large remaining indifferent.

Very few teachers have the courage of conviction to stand by their vision and care for children's growth in mathematics. Teachers, who are invited by Government Departments of Education, have mostly no professional status, not even membership in professional organizations. It is a matter for sorrow that professional organizations are rarely associated with the setups of Government Departments or Ministries of Education. Textbook writers have no experience of catering to the needs of the slow learners or demands of the fast learners. Publishers are more particular about getting the books accepted and turn over made commercially than about popularizing innovative ideas that work. Endowments are made for literary attainments and not mathematical attainments apart from allocation for top scores. Of late math contests are mushrooming and they are not diversified. There is no machinery to know what is going on in math classroom at

all levels. Since the assessment of work done by the teacher and the taught is through verbal performance in a written test, which again is based on a modal question paper, the schooling is geared to cater to this need that gets recognized as examination curriculum. It is not difficult to x-ray the situation if only one cares to interview, the so called centums, whose images of mathematics are as varied as one can imagine.

Though mathematics is the king of all arts and the queen of all sciences it is not thought of as part of school annual shows and so it is confined to classroom teaching alone. The cultural aspect of mathematics has not received the recognition it has in the universality of mankind.

In spite of the fact that man is living today in the platinum age of mathematics what with arithmetic becoming arithmetics, algebra-algebras, geometry-geometries, logic-logics, mathematics education is yet to cross the stone – age in teacher-pupil relationship and in getting the best out of all children and in access to books, videos and computers (CD's) which continue to come out in profusion though one has to be careful in selecting them as the mathematical background of producers cannot be just assumed to be relevant to developmental needs of children, expository needs of teachers and background needs of parents. Moreover they are costly and so shelved. Very few managements even think of them.

It has become fashionable to talk of math laboratories, but no B.Ed College, offers guidelines in raising them. How many B.Ed colleges have brought out publications regarding labs.

As a matter of fact, math lab will be very effective if only it forms part of time table and practical's paper is included in assessment. If responses to the practicals paper are non-verbal, one could see how much erroneous and coloured is the prevailing mode of assessment of children's attainments. Since it is non-verbal, children could be seen to perform 20 tasks in 1 hour as evidenced in the workshops. All this is not dreamy but tested and found working successfully. It is called polytechnic approach. Assessment should also include non-routine projects – recreational and enrichment extension to prescribed portions. Even the written tests in lower classes should not only be verbal but should carry visuals, percentage of coverages progressively declining with classes in the ladder of schooling.

Since children have to learn in English medium and since Mathematical communication centers round technical use of words, there is need to set apart, a part of the question paper for vocabulary.

There is a basic need to change the report card that mentions pass or fail in school examination. It would be ideal if it is kept in view in the public exanimation certificate of results as well.

What should the certificate indicate? Attainments and credits to announce the capacities and capabilities of the bearer.

It is forgotten that there are slow learners who shoot up in later life with contributions that have etched their names in the history of mathematics, science and technology. Whence come the day every examinee will be supplied

his/her answer papers? Should not the question paper form part of presentation of marks? There is no absoluteness in marks got.

If the school scenario were to see healthy changes that would recognise a student's success and not failures, particularly mathematics which is not a cup of tea for many, there should be awareness and effort to seek and study the openings to get the guidelines from all centres where mathematics is used or applied. No student at any level should be silenced when he or she raises a question or doubt even if it is, beyond the prescribed syllabus, or not known to the teacher. The report card sent home should carry among profile items the important item of curiosity level of the learner. How is curiosity level assessed one might ask, as it is not in vogue today?

If every school-going student is encouraged to maintain 'My Doubts book' showing doubts arising and recording at what stage and by what means they are in getting cleared and 'My Discoveries Book' showing records with dates of what are got by way of discovery. These can be examined for grading creativity level. Sometimes children think of ideas, which have escaped teachers! The teachers should be required to welcome them and quote them, if relevant, in regular teaching. Research attitude particularly in mathematics where it is more productive should be cultivated right from the day the schooling begins.

Curriculum transaction can therefore serve as a pace setter and an illuminating guide in helping teachers to know what they have missed and what they need to know in making themselves learner-friendly and progressive in educational thought by acquiring expertise in understanding alternative

strategies of instruction in respect of every curricular theme. Besides, teachers will have to be fully conversant with growth in conceptual understanding from what is considered as entrance level of exposure to exit level of elevation. Interested managements, parents and media reporters will find the references useful and helpful. A brief outline of what is being covered at primary level will be appropriate at this juncture.

Could anybody deny that there is continued stagnation not only in treatment of curriculum themes but also in what children are supplied with, the same foot rule and in the same set of geometrical instruments? Does this not expose the bankruptcy of educators in these days that witness changes in almost everything used at home and playground; leave alone in shopping and industries.

Given two foot rules, children can read off basic addition tables and tables of complements nine and ten compliments in particular. Not only that, they can do subtraction and get to see how multiplication arises.

The geometrical instruments provided continued to be not made use of fully. Only compasses, centimeters ruler and protractor are mostly used. An excellent handbook mostly on use of all the instruments in math learning is yet to see the light of the day.

Maximisation of use of limbs, fingers and toes in particular for concretization or visualization, setting up tables of addition and multiplication by way of reinforcement, decimal fractions and metric measures is yet to be thought of and resorted to by practicing teachers. When it comes to equipment for activities or experiment, mathematics is

least expensive. Every home and every class in school can afford to have Non Inventory Resource Materials Bank, consisting of card-board boxes with flat sides / faces, wooden blocks, bottle tops, one side printed paper, etc., Use of ruled sheet with and without margin, square dot sheet, triangular dot sheet and square ruled sheet is a necessity that is yet to be conceded by educators engaged in making mathematics tangible. As a matter of fact a math kit for hands on kick off take off learning experience has been made available by Ramanujan museum and math Education centre in 90's and they have received patronage. Since the curriculam people have not recommended their welcome, popularization is slow. It is drowned in the bustle for examination preparation schedule of schools, what with cycle tests, term tests and so on.

If one were to examine the primary math syllabus, to begin with, one could see abruptness in transition from arithmetic to algebra, from algebra to theoretical geometry and even within topics themselves. If the transition is smoothened through introduction of few concepts which though not familiar are quite simple, both the teachers and the taught find transactions breezy and exiting. In short, as has been observed in the conduct of workshops and projects, what is presented to promote hand-on experiences in concretization of concepts demonstrates amply take-off reached by almost all children in abstract thinking and generalizing. To clinch the sharpness, use of example and non-example in respect of definition and example and counter example in respect of statement is found very telling and testing. One could witness coverage of more mathematics in less time and less drill promoting learning for permanence. When mathematics is the order of the day in all spheres of corporate life, schools will be failing in

their responsibility of preparing the wards sent to them, if they do not provide through compound wall paintings, bulletin boards and show-cases materials that attract and provoke thinking and discussion and cultivating of mathematical eye. Let us remember that 'what eyes see not, the heart aches for not'. No school can boast of providing a favorable climate for appreciation of mathematics and its image, if there are no mathematical portraits of break-though mathematicians like Gauss, Euler and Cantor, Galois, Ramanujan. The simplest, least expensive and non-time consuming way to mark their birthdays is to put up a biographical note with commemorative date magic square prepared by children. As construction of date magic square involves only addition and subtraction and placement with no repetition, children vie with each other in fixing up the birth date magic square. Internet makes the exercise easier today to finding the dates of birth of eminent mathematicians.

No academic year should miss holding three-day three-tier math exposition for walk-across the curriculum to develop receptivity of expectancy.

There should be every attempt to build up the image of mathematics as a study of patterns, relations and structures, which even the research mathematicians entertain. The practice of treating mathematics as one of the subjects may be relevant from examination schedule but not in the attitude of teachers. The entire staff should show interest in mathematical core of every subject. If teachers were to be facilitators and stage-setters and not directors in learning process neither bemoaning or sloganeering nor gubernatorial exhortations will be enough. The furniture in the classroom should get first changed, the long desks and

benches giving place to dual desks and benches for children to move about, do group work under supervision by teachers going from one spot to another. The blackboard space should be increased. As a matter of fact, black boards, rather green boards, to reflect the changes today should be available in three or four places in the classroom for simultaneous expression and interactive learning by students. Just as the difference between haves and have-nots are being narrowed in society, the difference between copes and cope-nots need to be reduced by facilities to work in group with assistance from group leaders. The climate that sees children learn only what is taught should change to children learning allotted portions on their own. Even the regimen of *mental sums* and *homework* should give place to *warning up* and *follow up* that are broad based and preparatory. Every attempt needs to be thought out and made to increase receptivity of learners. This will reduce the spell of T.V. programs and other coloufully and pictorially produced periodicals and books.

The school administration will do a creditable job if it could set apart in their annual schedule of work days for disciplinary meet and inter disciplinary meet of teachers, the contributions of teachers going into the records of their service files. This can be seen to save teachers from stagnation resulting from repetition year after year and inducing them to update and upgrade their professional background through study of books and periodicals of professional interest. Unless every opportunity is looked for and availed of, standards of mathematics learning and teaching will not be raised much to the detriment of the future of children and dignity of the nation in the international arena.

Our rich mathematical heritage and our ancient cultural monuments and traditional customs like kolam and rangoli beckon us to adopt all means through all media in popularizing mathematics in order to make math phobia giving place to math mania and math philia. It is not that everyone should become a mathematician but appreciation of mathematics should be as wide spread as that of art, music and dance.

(1) NUMBER SENSE

Once the new born begins to speak, grand parents and parents hasten to show how smart they are by making them recount 1 to 100 and getting their praises from visiting relatives, not knowing that harm is being done to development of number sense in them.

Animals and birds have number sense and they don't count. Reciting counting names in their order exhibits ability to give a sequence of sounds without the number notion that requires 'one more' and one less' concepts.

As a matter of fact, if only the breast feeding mother is on the look out for some days the breast fed child could be seen to place its hand one day on the other breast while sucking on one side. To start with, the mother does changing its side and gradually it dawns on the baby that there is the other breast too for suckling. That moment needs to be noted and celebrated with invitation cards and party. All along, the biological events have come to be celebrated by families with fervour, pomp and show. It is high time we take note of intellectual events and celebrate them as well, leading to and promoting culturalisation of mathematics, significantly not noticeable today. Number sense centers round the concepts of discrete and continuous, *one and the other* (giving rise to recognition of two ness) *one and another*, *one by one* (giving rise to 'one more' and 'one less', 'successor' and 'predecessor' concepts) *one and many* (giving rise to different kinds of many ness), as many as (giving rise to one to one correspondence, sets having the same number of things leading to recognition of number.)

Children are encouraged to say 'tata' and 'bye'. They can be encouraged to put fingers in one hand, match finger in the other hand (Leading to more and less finger with some folded) and get introduced to five through five fingers. Children know one, two and five and four and finally three. Their comprehension can be elicited by inviting them to demonstrate. Show me one head, one nose, one chin, one neck, etc. show me two legs, two ears, two hands, two legs, etc. show me five fingers, five toes.

Children watch birds flying with two wings, and strutting about in two legs and animals walking about with four legs. Children can be asked to walk on fours. These are exposures which need to be noted through casual conversation for a few minutes as and when occasions come.

Some children learn by imitation, parents and elders, for that matter, may better, for the sake of the growing child, talk among themselves by way of enacting. This quickens the pace of acquiring number sense and number recognition.

Three can be introduced by showing the blades of a fan.

Once children become familiar with the numbers 1 to 5, they are ready to be tested informally by changing roles, which children like. Ask a child to call for showing a certain number of fingers, say three fingers. When you show two fingers, the child says 'no' and puts up one more finger of yours. When you show four fingers, the child says 'no' and folds one of your fingers shown.

This test is an excellent one. When both the palms are used in exhibiting fingers up to five, readiness to do addition,

subtraction, multiplication and division gets cultivated very informally, leading to receptivity to such situations later.

Having learnt numbers 1 to 5 *not in order*, order can be taught subsequently, show one finger stretched, the rest remaining folded in one hand and ask the child to tell you what should be done to show two fingers. The child will raise one of the folded fingers and put it up. It is time child is exposed to the communication underlying behind this exposure as one and *one more* is two. Over days, 1 to 5 can be given out in order leading to 1, (one more) 2, (one more) 3, (one more) 4, (one more) 5. Ask the child to give out the numbers without saying 'one more'. The child says 1,2,3,4,5.

Show two fingers and touch them one by one, say, 1, 2. The child makes the discovery that when objects are touched (or considered) while saying 1,2, etc in order, the number of objects is the number where you stop. This is a very great discovery for a two and half year old child.

An intelligent child can be seen to ask 'what is five and one more'? Six gets introduced. This way nine and one more is ten (not one zero) is learnt. This is what is expected of parents and teachers in play school or lower kindergarten.

That 'every time one more is said a greater number is identified' is a mathematical representation which no child should be denied through short circuiting practice of premature counting practice.

Now children are ready to get at the concept of zero. It is very unfortunate that zero is taught as nothing, that has been found to create a mental block in learning with

understanding the role of zero as the count or the cardinality of the empty set.

Instead of taking a collection of objects, removing the objects one by one and finally saying nothing or no objects, it is more meaningful and operative to face situation when absence of objects occurs in one place and presence of objects in other place or places.

Show, say, four finger in one hand and three fingers in the other hand. Ask the child to give the numbers. The child can be seen to say 4,3. Now fold one finger in one hand, say the hand showing lesser number of fingers (greater number of finger showing hand can also be considered). Ask the child to find the number. The child is seen to say 4,2.

Continue folding the finger, one by one, keeping the four finger hand unchanged. The child is seen to say 4,2; 4,1; 4, no finger or nothing.

Step in to help the child communicate that there are zero fingers, that is to say 4,0.

Now continue folding the fingers in 4 finger hand one by one, allowing the child to give the number. The child is seen to say

3,0; 2,0; 1,0 and 0,0

So '0' is seen to be associated relatively. Otherwise 0 as nothing will be meaning absolute nothing, which block conceptual clarity.

Alternatively start folding one by one fingers in the 4 finger showing hand, keeping 3 finger showing hand unchanged till all the 4 fingers get folded. Now the child gets to see 4,3; 3,3; 2,3; 1,3; 0,3; 0,2; 0,1; 0,0.

You can keep in more than one place (or plate or box) some objects and by removal of objects one by one or more, children can be asked to give the numbers in the place.

As one of the more reliable ways of testing is to see if the concepts learnt are applied in mere context not made familiar earlier, a child at this stage of knowing the siblings he/she has or the number of boys or girls in a family photo and so on.

Two boys family	zero girls.
Three girls family	zero boys.

Some children out of ecstatic understanding can be seen to real out: zero cats, zero dogs, zero crows, etc in the classroom. If such an event were to take place, the child should be applauded by the class, clapping hands (of course in a disciplined manner inculcated by practice).

Children can be asked to give more examples to encourage observation, understanding and expression.

E.g.

A cat has zero horns.
My grandmother has zero teeth.
I have zero tails and so on.

Now, what is 10? It can be seen that educated adults reading it as ten without bothering about the roles of 1 and 0 in it.

10 is ten only when the basic of counting is ten. When bundles of ten are made out of sticks say, and when 1 bundle alone is taken and there are no loose sticks, then it mean 1 ten and 0 ones, giving 10 (one zero) meaning 10.

If there are bundles of fives, taking 1 bundle mean 10 (one zero) giving 1 five and 0 ones.

So 10 may mean any number greater than1, if that number is the basis of counting. It shocks many to know that 10 (one zero) may mean one hundred if bundling is in hundreds.

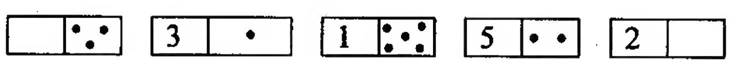
What does 1000 mean?

One thousand, 0 hundred, 0 ten, 0 ones. It is not enough if children are taught to recognise it as simply one thousand.

§ *Numerals:*

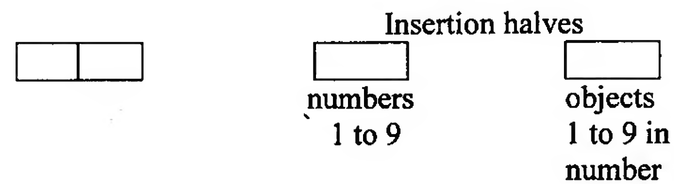
Teaching numerals 0 to 9 needs to be done interestingly and impressively. Usually dominoes are used to set up domino chain.

E.g.



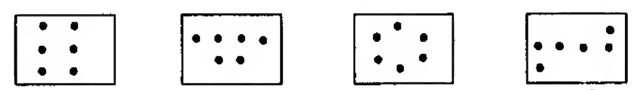
This practice is seen in play schools. But there is one disadvantage. Once the child learns to set up the chain,

then memory takes over and the child will do it again with or without understanding. To avoid that undesirable situation, the device of split domino chain can be resorted with tremendous effect. Keep insertion strips ready with as many as required for insertion strips ready with as many as required for insertion of cards from either side. Let the item to be tested be split and now the parent or teacher can set up dominoes to form a chain that is not the same all the time.



Development of number sense through arranging objects in rows in different ways is an excellent avenue for getting to know the number intensively and giving the number by inspection.

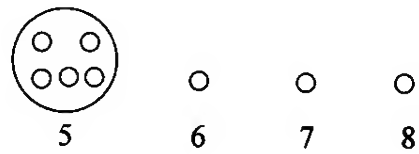
E.g.

Six 

and so on.

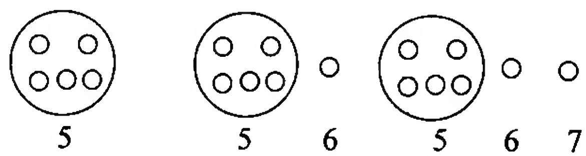
§ *Evaluation:*

The child exposed thus will be able to pick out objects upto five without counting and set up objects greater than five in number by taking out five first and then the required number by adding on by saying, say, five, Six, seven, eight. In case of eight



Given objects more than five, children can have valuable experience by being asked to make the number through reading

E.g. From 5 to 7

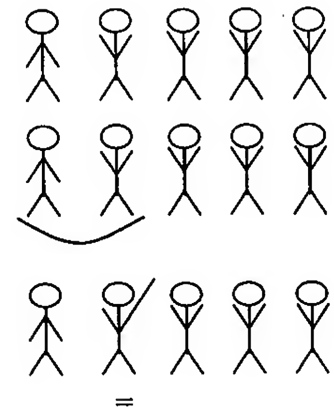


A start needs to be made to expose children to distinguish between numbers used in cardinal (how many) and ordinal (which one) senses.

Three

A child can hold three fingers together from thumb or little finger and say three fingers. The child can show the third or middle finger to say *third.*

This can be had on the play ground as well or children facing the class in a row.



Five children

Two

Second

At home also, parent can display bottles and ask for the second bottle from the left and then to take out three bottles and set them apart. This can be practiced with a bunch of bananas, pointing the second banana and plucking out two bananas and so on.

This experience will get enforced if on looker children are asked to give 'how many' and 'which one', when children of some number step forward and someone raises his/her hand or put on the cup. If more wear caps the question is how many wear caps now and if one of the cap wearing child is to be called, the question is which one.

Note: Remember always that concepts and generalizations are got at by *Multi Embodiment principle* (MEP) and Multipurpose use of objects (MPU).

(2) NUMBER BEHAVIOUR

Number behaviour fascinates everybody once one gets inclined or introduced to it. To start with, certain behavioural patterns need to be familiar to every lower primary child.

When is a number even! There is a universal tendency to state it by the last digit of its numeral being even, that is to say 0,2,4,6 and 8.

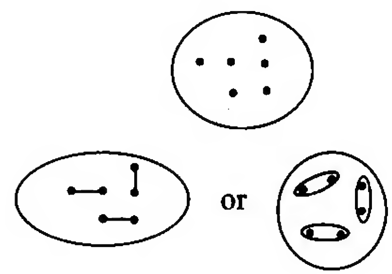
If asked how these are even, the routine answer is, when any of them is divided by 2, the remainder is zero.

Care is not shown in distinguishing between numbers and numeral. A number is abstract and is expressed by a number name and a numeral (symbol), which again is related to the base of numeration.

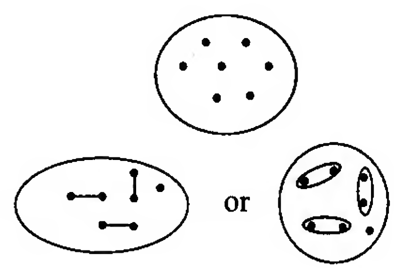
When one is asked about the sum of two odd numbers, the practice almost universal is to take two or three cases of pairs of odd numbers, find their sum and jump to the conclusion that it is even. This is an attitude, which is universal to mathematical competence, as one cannot generalize by means of finite number of examples. What is true of them is not here always. Consider for example $\frac{16}{64}$ cancel 6, what do you get is $\frac{1}{4}$ which is the reduced from of $\frac{16}{64}$. Consider more examples. $\frac{166}{664}, \frac{1666}{6664}\ \frac{16666}{66664}$ and so on.

It works out that way. But the cancellation is not acceptable as only a common factor can be cancelled.

One may wonder how a child can get the general property. Give a child some seeds without being counted and ask the child to pair them. When the seeds can be grouped into pairs perfectly, the number of the seeds is even. When, one is left, every time it is so when odd number of objects is there, the number of objects is odd. If need be the seeds can be counted and then the number thus got is considered even. For practice, certain number of dots can be taken and through pairing the number of dots declared even or odd.



The number is even on counting, them it is six. So six is even.



One is left out. So the number is odd. On counting, the number is 7. So 7 is odd.

It is a discovery and an experience for children to realize that whenever objects in a collection are paired either 1 is left out or nothing is left out.

This approach enables even a child not knowing number of digits exceeding one to be in a position to give the odd or

even behaviour of a collection and odd or even behaviour of the sum of two or more collections.

even + even = even (see all the objects are in pairs)
even + odd = odd (see one object alone will be left)
odd + odd = even (see one left out of the first collected together with one left out of the second collecting will form a pair and since all are in pair). The number of the two collections put together is even.

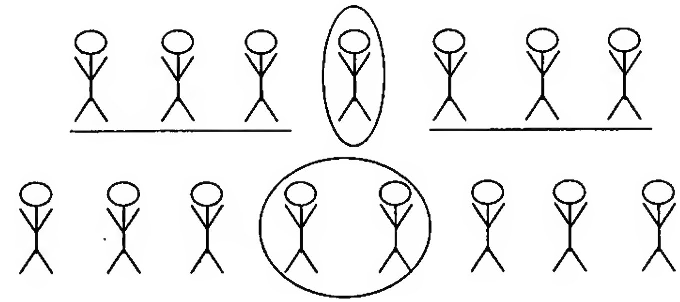
On the playground also, this experience can be had. Ask children to be moving about giving a whistle and instruct them to form two groups. Let children in each group go into pairs holding hands. When in a group, there is one without a pair to hold hands, that group has odd number of children. Otherwise they have even number of children.

This can be continued with a tremendous drive for exercise, imagination and intuition.

E.g.
odd + odd + odd = odd (Since odd + odd is even and so even + odd = odd).
As these do not relate to particular number, the conclusion spelt out is general.

Multiple and factor:

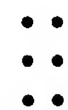
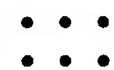
Set up two rows of children to observe that, when the number of children or objects in a row is odd, there is a middle one with equal number on either side. If the number of children is even, there is no middle one or two children can be identified to form a middle pair, equal number on either side.



On the other hand,

Giving the child, say, six objects and instruct the child to place them in an array showing rows of equal number of objects.

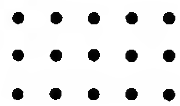
What the child gets is

Either  or 

6 objects are placed in three rows of two each or two rows of three each. Now comes the communication.

6 is a multiple of 3
6 is a multiple of 2
2 is a factor of 6
3 is a factor of 6

Give the child an array of dots

say 

How many dots are there? 15
How many rows? 3
How many in each row? 5
So 3 and 5 are factors of 15 and
15 is a multiple of 3 or 5.

This experience can be had in the playground as well. Ask some children to make an array. Once array is formed ask the participant children and onlooker children to give the number of rows and the number in each row besides the number of children forming the array.

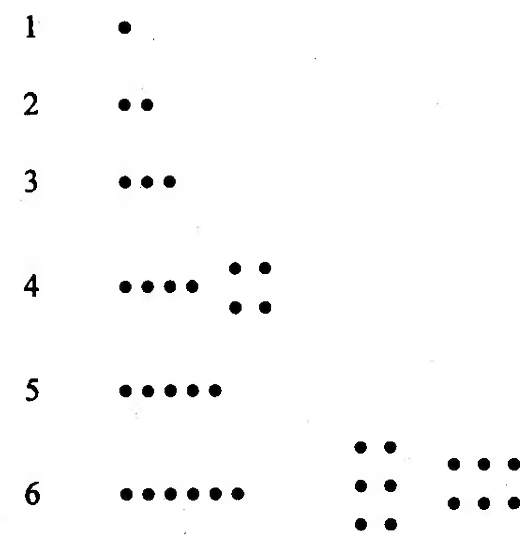
Now the number of the array is a multiple of number of rows as well as number in each row. Looked at in another way number in each row or the number of rows making up the array - are the factors.

Children can also see their stretched palms and see the marks at the places where there are 6 ones for folding partially or fully. By keeping two, three or four finger other than the thumb stretched, children can see two rows of three each, three rows of three each, four rows of two each, four rows of three each. If tips of fingers are taken into consideration, one can have two rows of four, four rows of three, etc.

Children can take broomsticks or tongue cleaner and by placing this criss cross, they can see more arrays and use them to give multiples and factors.

In tilak (Bindi) sticker cards, sold and bought by their elders, children could see this situation.
Once children become familiar with array and rows leading to the concepts, factor and multiple, they have the readiness to take up situation leading to the concept of prime and composite number.

Give children some buttons not exceeding ten to start with. Let children perform the experiment of taking them in counts of 1,2,3, etc and checking if for each count they can be pleased in a row and then in an array.

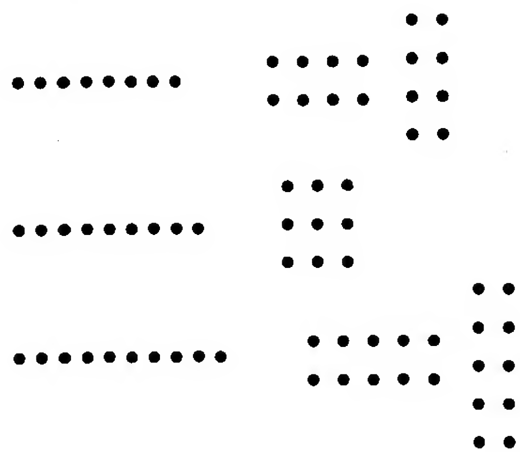


7

8

9

10



What do they discover?

Any number of buttons can be placed in a row. But only some number of buttons can be placed in a row as well as in an array. Ask children to record their findings by means of dots.

At this stage children are ready to distinguish number as prime and composite.

If objects can be had in a row alone, the number of objects is *prime*. If besides, they can be had in an array, the number of objects is composite.

What about 1?

If an array were to have rows, a row should have more than one, in which case 1 is neither prime nor composite.

This experience can be had on the ground with groups of children, checking this number behaviour.

Having sufficient exposure, ask children to form a table to record the number, give the factor of each.

Number	Factor
1	1
2	1,2
3	1,3
4	1,2,4
5	1,5
6	1,2,3,6
7	1,7
8	1,2,4,8
9	1,3,9
10	1,2,5,10

What do children discover?

Prime numbers have each only 2 factors where as composite numbers have each more than two factors. Since 1 has only one factor, 1 is neither prime nor composite.

If children have been introduced to division, children can be helped to say every number has itself and 1 as divisor whereas some numbers have factors as well. Every divisor is a factor but not the other way. Strictly speaking children could also see that a prime number has two and only two distinct divisors whereas a composite number has more than two distinct divisors.

By way of investigation children, say, in class 2 or 3, can be asked to investigate by experience with real objects not exceeding 20, the nature of numbers beyond 10.

Children in higher classes could also explore the nature of the sum of two prime numbers, two composite numbers, one prime and the other composite. Excitement is felt when guesses are to be made.

This is a group activity or classroom interaction session. There is a lot of excitement in that they find their assumption not being valued.

prime + prime. Is it prime or composite? Suppose it is given prime, then

3 + 5 = 8, showing prime + prime is not prime.
But 2 + 5 = 7, showing prime + prime is prime.
So prime + prime can be prime or composite.

The vocabulary of counter example can be introduced with regard to statement and non example with regard to description or definition in classes from five onwards.

What about composite + composite?

Take 4 + 9 = 13, it is not composite, but in
6 + 4 = 10, it is a composite number.

As in the case of prime, the sum of two composite numbers may be composite or prime.

What about composite + prime?

Take the example 6 + 7 = 13, 13 is not a composite number.

But 8 + 2 = 10, where 10 is not a prime number.

Through such experiences children can take the first step in being careful about describing mathematical behaviour leading to mathematical maturity.

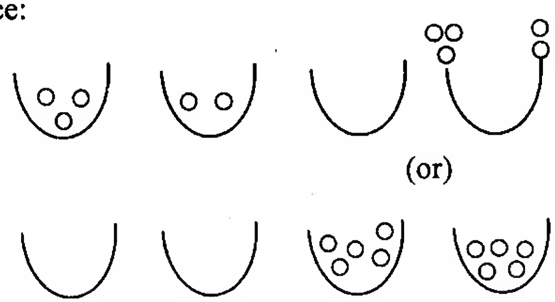
(3) LIFE SITUATION APPROACH TO OPERATION OF ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION

Traditionally, the practice has been to introduce children to skills in performing operation and then doing story problems. Since children are not motivated in knowing the need for the basic operational skills, they find the experience dull and heavy. When they reach the stage of doing word problems or story problems, children look for cue words or cues from teachers regarding the operation to be chosen. Textbooks could be seen presenting the skill application approach. There is no clear direction in the prescribed curriculum either.

Situations giving rise to the basic operation need to be dramatized through pictures and demonstrated in the field. At home, parents could point out appropriate situation arising at the dining table, kitchen, washing and drying clothes, sleeping place, etc.,

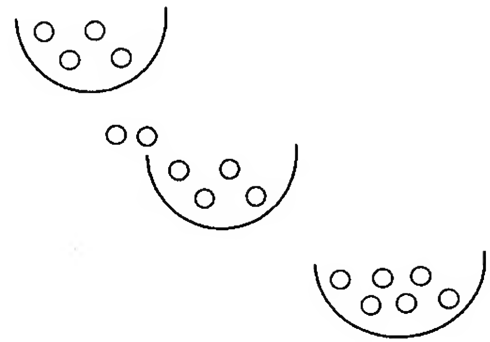
ADDITION:

Putting together and increasing.
For instance:



3 + 2 = 5

Increasing situation:



4+2 = 6 (four and two make six).

Dramatisation: (Together situation) two groups of children come from two different directions, and form a group. There is already a group.

(Increasing situation): some more join the group.

Vocabulary: on the whole, how many altogether.

(Warning: Teaching by cue lands a learner in confusion. For e.g. Give this sum and see what the response is. 8 cups of vanilla ice creams are taken away from the fridge and 2 cups of pista ice creams are taken away from the fridge. How many are taken away? Children who go by cue are seen to get the answer 6 as ‘taken away’ is used only in subtraction).

Subtraction:

Subtraction arises in situation requiring

(i) Taking away (removed, spoiled, dropped, etc)

(ii) Comparison (finding which is greater or smaller, by how many – there is no taking away),

(iii) Complementary addition (finding the required number to get the expected number).
There are two more situations:

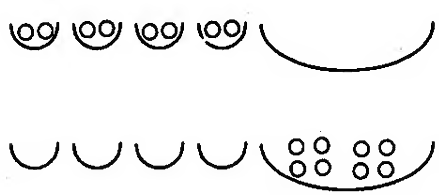
(iv) Finding the number what should have been there form what is there and what is included.

(v) Finding what is taken along given the outcome and the number taken at the beginning.

Multiplication:

This arises out of their life situations:

(i) Collection of packages with equal number of items and finding the total number of items (Repetitive addition)

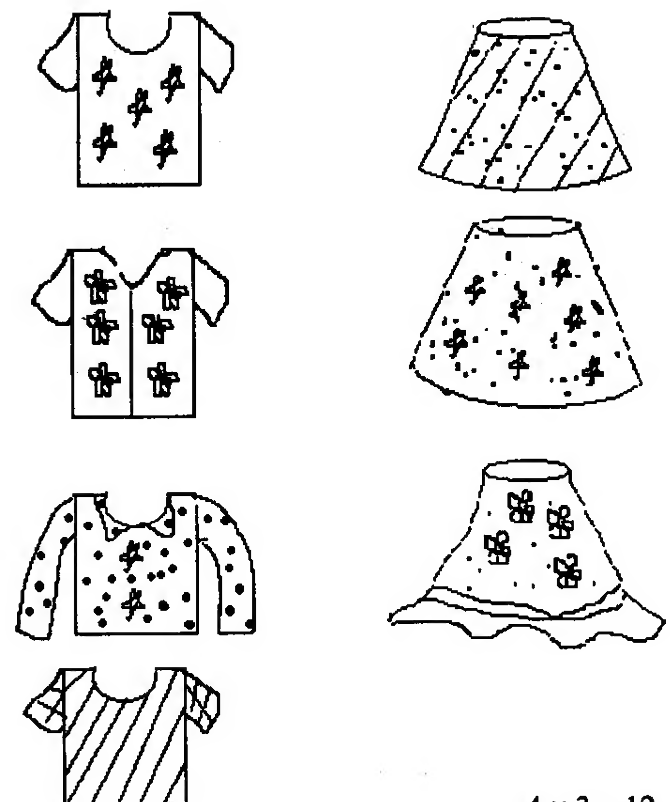


2+2+2+2

$4 \times 2 = 8$

(ii) Selecting required number from the display of group with equal number of objects (Times situation).

(iii) *Cartesian product or pairing:*
Given two kinds of things and finding the number got by pairing things of the first kind with things of the second kind
Four shirts and three skirts: A shirt and a skirt make a dress. How many kinds of dresses can be had?



$4 \times 3 = 12$.

Note: It is surprising that such an interesting item is excluded from all curriculum, CBSE, Matric, etc.

Division:

As multiplication ⬭ means repeated addition of the same number, division means repeated subtraction of the same number for a given number leading to division process or algorithm.

To begin with division arises out of

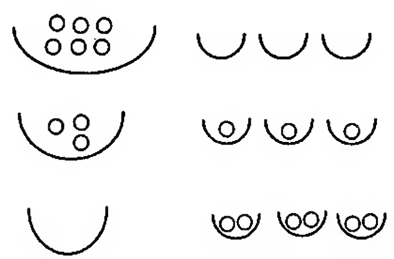
(i) Equal distribution among others or among themselves.

(ii) Equal grouping

(iii) Complementary multiplication.

(iv) Repeated subtraction.

(i) Equal distribution requires knowing what is got by each person or what is got in each container.



Distribution of six apples among three baskets given 2 apples per basket.

Though distribution of six sweets by 3 persons themselves, each gets 2 sweets.
Children should be helped to know that in distribution, rate is involved.

$$\frac{30 \ \textit{mangoes}}{6 \ \textit{persons}} = 5 \text{ mangoes per person.}$$

$$\frac{40 \ \textit{rupees}}{2 \ \textit{shirts}} = 20 \text{ rupees per shirt and so on.}$$

It is important that children get to associate 'per' with division, giving them the frame of mind to do word problems in division.

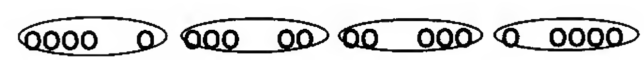
(ii) *Equal grouping:*

Finding the number of groups when certain things are grouped equally.

0000 0000 0000 0000 0000

There are 20 children. They are to keep together in groups under the care of a teacher for each group. If in each group there are to be 4 children. How many groups will be formed?

There will be 5 groups. On the other hand if there are to be five in each groups, how many groups get formed.



Four groups will be formed.

$$\frac{20 \quad children}{4 \quad children} = 5$$

$$\frac{20 \quad children}{5 \quad children} = 4$$

Also,

$$\frac{20 \quad children}{4 \quad children \quad per \quad teacher} = 5 \text{ teachers}$$

$$\frac{20 \quad children}{5 \quad children \quad per \quad teacher} = 4 \text{ teachers}$$

$$\frac{200 \quad tablets}{20 \quad tablets \quad per \quad strip} = 10 \text{ strips}$$

Note: Speed is given in km per hour. If division is introduced the way pointed out here, speed can be taken to mean rate obtained by dividing distance in km by time or duration in hours.

Complementation:

Nine complement and ten complement makes subtraction 'easy'. For example consider finding the difference between

(i) 1000 and 256, the difference can be just given without 'borrowing' as follows:
Nine complement of 2, nine complement of 5 and ten complement of 6 = 744.

(ii) 10000 and 608.

The difference:

608 is to be taken as 0608. Nine complement of 0, nine complement of 6, nine complement of 0 and ten complement of 8.
= 9492.

Note: The difference is written from the left instead of from the right as in borrowing method.
This is considered as Vedic Mathematics.

Alternatively, complementation can be used in another way. Find the difference

$$\begin{array}{r} 52768 \\ \underline{14879} \\ \underline{37889} \end{array}$$

ten complement of 9 is 1 1+8 = 9
nine complement of 7 is 2 2+6 = 8
nine complement of 8 is 1 1+7 = 8
nine complement of 4 is 5 5+2 = 7
nine complement of 1 is 8 8+5 = 13, drop 1, write 3.

Rationale:

52768 – 14879
= 52768 – (100000-85121)
= 52768 + 85121 – 100000
= 137889 – 100000
= 37889.

(iv) Repeated subtraction:

Each winner in a competition is to be given gift cheque of Rs. 25 each. The amount to be spent is Rs.100. four competitions are held. In the first competition, there is one winner. He is given a gift cheque. In the second competition, there are two winners. Each of them are given a gift cheque.

In the third competition, there is a winner and he is given a gift cheque. How is this accounted?

	I	II	III	
Rs. 100	–25	–25	–25	
		–25		
				0

Through repeating subtraction it is seen that allotted money is spent, having no amount.

$$100-25-25-25-25 = 0$$

(Repeat subtraction)

$$\begin{array}{r} 25)\ 100\ (4 \\ \underline{100} \\ 0 \end{array}$$

Note: Division process or algorithm is the repeated subtraction.

(4) TABLES OF ADDITION AND MULTIPLICATION

There has been the long established tradition of remembering tables by recitation and rote memory Many children have suffered ego damage resulting in self-diffidence and dislike of mathematics.

Tables come to mean multiplication tables and you may ask yourself or anyone who has had the schooling to state the number of tables learnt by rote. Invariably the answers are not uniform and they centre round 'tables up to 12 times' 'tables up to 16 times', 'tables up to 20 times'. Sometimes tables up to 12 times each reaching to 20 times. Books of tables up to 12 times each reaching to 20 times. Books of tables are sold all over the country depicting this situation. Tables books have no picture.

A test will expose the harmfulness of this practice. I have often given at the beginning of the workshop for teachers, questions like the following:

Add	Add
3×8	7×9
5×8	4×9
2×8	9

Find the difference:

9×5	8×6
6×5	7×6

The working has been as displayed below:

3×8	24	7×9	63
5×8	40	4×9	36
2×8	16	9	9
	80		108
9×5	45	8×6	48
6×5	-35	7×6	-42
	15		6

On the other hand, how should they have done if they have comprehended fully the meaning of multiplication?

Addition

$$\begin{array}{l} 3 \times 8 \\ 5 \times 8 \\ \underline{2 \times 8} \\ \underline{10 \times 8} = 80 \end{array}$$

The question that is to be asked should be 'How many 8's are there?' 3+5+2 or 10 eights. Thus, the sum is 80.

$$\begin{array}{r} 7 \times 9 \\ 4 \times 9 \\ \underline{9} \\ \underline{12 \times 9} = 108 \end{array}$$

How many 9's are there? 7+4+1 or 12 nines. (9 means one nine only). This is not recognised to start with.

SUBTRACTION:

$$\begin{array}{l} 9 \times 5 \\ \underline{6 \times 5} \\ \underline{3 \times 5} = 15 \end{array}$$

How many 5's are there? Nine.
How many 5's are to be taken away? Six.
How 5's are left? 9-6 = 3
So the difference is three 5's or 15.

$$\begin{array}{r} 8 \times 6 \\ \underline{7 \times 6} \\ \underline{6} \end{array}$$

How many 6's are there? Eight.
How many 6's are to be taken away? Seven.
How many 6's are left?
So the difference is one 6 or 6.

Sometimes deplorable responses have been noticed such as the following:-

3×8	7×9
5×8	4×9
2×8	9
$10 \times 24 = 240$	$11 \times 27 = 297$

$$\begin{array}{l} 9 \times 5 \\ 6 \times 5 \\ 3 \times 0 = 0 \end{array}$$

First of all there should be distinction between basic tables and tables.

Basic tables refer to digits 0 to 9 in base ten numeration system, whereas tables of any number can be given to any stage in each.

'Why are then the tables taught the way they were and they are' is an issue that has to be understood first. During the pre independence days when English education took root, 1 Re had 16 annas and 1 anna 12 paise in India and 1 £ equalled 20 shilling, 1 shilling equaled 12 pence. Since non-metric system of measures were in use, basic tables were neither taught nor learnt as such.

BASIC ADDITION TABLE:

They relate to 100 basic addition facts relating to digits 0 to 9.

They can be built by children themselves in different ways. Intuitively-better-endowed children learn the tables fast, since slow learness require special care with approaches suited to their make up, then there should be choice for constructing them. Some are explained below:

1. *Addition 1 table:*

	1	2	3	4	5	6	7	8	9
	•	••	•• •	•• ••	•••••	••••• •	••••• ••	••••• •••	••••• ••••
+1	•	•	•	•	•	•	•	•	•
Sum	2	3	4	5	6	7	8	9	10

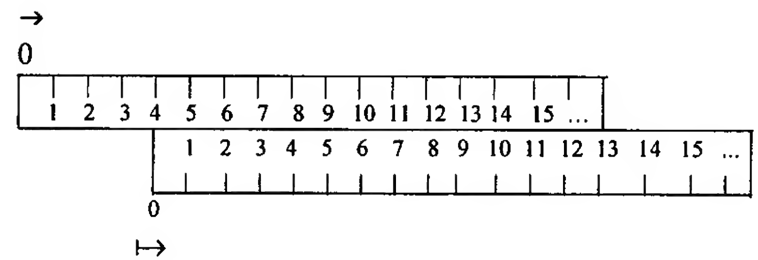
Addition 2 table:

+2	••	••	••	••	••	••	••	••	••
Sum	3	4	5	6	7	8	9	10	11

and so on for other digit tables.

2. *Use of two foot rulers:*

By placing one foot ruler against the other in such a way that the number to be added is placed below zero, then addition number tables can be read off and by familiarity of practicing thus, basic addition tables can be construced, written out and remembered. Addition 4 table is pictured below for instance.



$4 + 1 = 5$

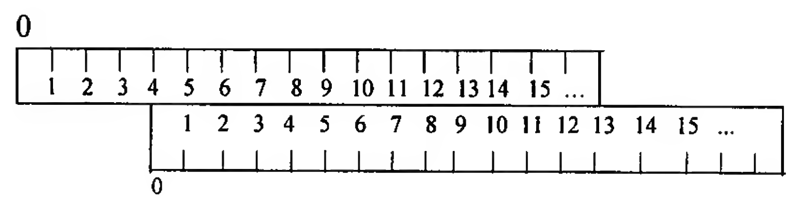
$4 + 2 = 6$ and so on up to $4 + 11 = 15$, by reading off to the right.

3. Two check ruled sheets can also be used for movement to read off basic addition facts.

Note: by placing 0 of the sliding ruler or strip below the 0 of the fixed ruler or strip, addition facts of zero can be read off as

$0 + 0 = 0$
$0 + 1 = 1$
$0 + 2 = 2$, and so on.

Incidentally, difference can also be read off using two rulers or strips, one fixed and the other sliding as pictures below:



To get, say, 9-5, bring 5 of the sliding ruler below 9 of the fixed ruler in alignment. Read backwards, go to the left and read at 0 of the sliding ruler or strips.

What is read here is 4 which is the difference between 9 and 5. So 9-5 = 4.

Incidentally, other subtraction facts with the same difference 4 can be read off:

$13-9=4$	$10-6=4$	$7-3=4$	$4-0=4$
$12-8=4$	$9-5=4$	$6-2=4$	
$13-9=4$	$8-4=4$	$5-1=4$	

Note: some gifted children can be seen to spell out a pattern in these same difference situations.

When the same number is subtracted from minuend and subtrahend the difference remains unchanged. Also, when the same number is added to the minuend and subtrahend, the difference remains unchanged.

This empowers them to do some two digit number subtraction in an easy and elegant way.

$$\begin{array}{r} 34 \\ -\,27 \\ \hline \end{array} \qquad \begin{array}{r} 34+3 \\ 27+3 \\ \hline \end{array} \qquad \begin{array}{r} 37 \\ -\,30 \\ \hline 7 \end{array}$$

Another way:

$$\begin{array}{r} 50 \\ -19 \end{array} \qquad \begin{array}{r} 49+1 \\ -19 \\ \hline 30+1 \end{array}$$

It is not common to construct and remember complementary addition facts tables. Of them 10 complements, 9 complements and 5 complements tables are remembered.

§ One way is to take a collection of, say five objects and by setting apart the objects one by one, complementary addition facts relating to 5 can be constructed and remembered.

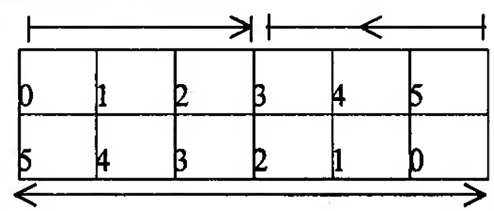
OOOOO		$5+0=5$
OOOO	O	$4+1=5$
OOO	OO	$3+2=5$
OO	OOO	$2+3=5$
O	OOOO	$1+4=5$
	OOOOO	$0+5=5$

a strip folded in such a way as to show 5 equal parts can be used to give the complementry parts relating to 5 by progressive folding and noting the number in the folder part and the number in the unfolded part.

1	2	3	4	5

1+ 4 = 5
2+ 3 = 5
3+ 2 = 5
4+ 1 = 5

Using two foot rulers or six inch foot rulers placed to match in opposite directions 5 – complement facts can be read off as seen below.



$5+0=5$
$4+1=5$
$3+2=5$
$2+3=5$
$1+4=5$
$0+5=5$

Note: children attention should be drawn in counting of equal intervals and not of separating lines or points.

Similarly using the two foot rulers and placing them in opposite ways in alignment, 9 complement and 10 complement facts can be read off and noted and remembered.

			0 1	2	3	4	5	6	7	8	9	10	11
12	11	10	9 8	7	6	5	4	3	2	1			

9 complements are nine in number and 10 complements are 10 in number.

§ Practice for this can be had on the playground itself with the required number of children 5, 9 or 10 standing in a row. By requiring children to step aside in groups of 1,2 etc one step to the left (or one step to the right), the complements for the numbers of children forming the row can be spelled out.

§ Practice can be had with *five* fingers, *ten* fingers and finally with *nine* fingers, with *one* finger folded.

§ The stage is set for skip counting, which creates readiness to fix multiplication table.

In twos:

1.	3.	5.	7.	9.	11.	13.	15.	17.
2.	4.	6.	8.	10.	12.	14.	16.	18.

In threes:

1.	4.	7.	10.	13.	16.	19.	22.	25.
2.	5.	8.	11.	14.	17.	20.	23.	26.
3.	6.	9.	12.	15.	18.	21.	24.	27

After reaching skip counting in fives multiplication tables can be introduced. First of all the teacher should ensure the meaning of multiplication as repeated addition of the same number, through examples and counter examples.

$3 + 3 + 3 + 3$ repeated addition.

$3 + 3 + 4 + 5$ not repeated addition, though the digit 3 alone is repeated.

$2 + 2 + 2 + 1$ not repeated addition, though the digit 2 alone is repeated.

These are some of the interesting ways of helping children build the basic multiplication tables.

§ Give each child a square ruled sheet 9 × 9 and buttons box. Ask the child to place the buttons in each of the squares in the square ruled sheet showing two rows.

	1	2	3	4	5	6	7	8	9
2 ×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗
	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗
	2	4	6	8	10	12	14	16	18

Help the child to use the placement to spell out 2 times basic multiplication table.

$2 \times 1 = 2$	$2 \times 4 = 8$	$2 \times 7 = 14$
$2 \times 2 = 4$	$2 \times 5 = 10$	$2 \times 8 = 16$
$2 \times 3 = 6$	$2 \times 6 = 12$	$2 \times 9 = 18$

and so on, up to 9 times table.

§ How to read the table? There are certain conventional ways.

(1) Two 1's make 2
Two 2's make 4
Two 3's make 6 and so on.

(2) Two one's are 2
Two two's are 4
Two three's are 6 and so on.

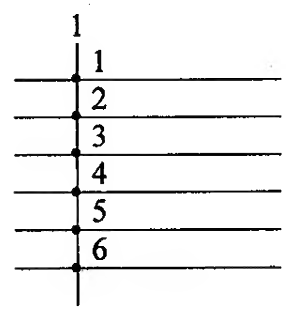
(3) Two times one is 2
Two times 2 is 4
Two times 3 is 6 and so on.

(4) 1 two is 2
2 two are 4
3 two are 6 and so on.

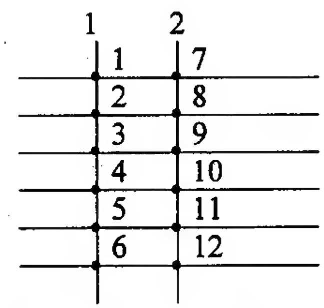
This distinction of 2×3 and 3×2 in recalling table 2 multiplication is to be allowed till children come to realize that 2×3 is the same as 3×2, otherwise described as the commutative property of multiplication. It is similar to commutative property of addition commutative in the sense that order of taking the two numbers does not matter. In other words, the result is the same in whatever order the two numbers are taken.

§ Construction through Criss-Cross placement of sticks or strips.

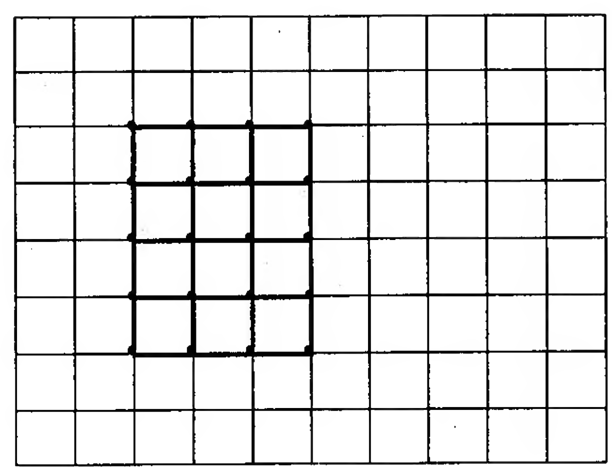
From 5 times onwards. Basic multiplication tables for 6,7,8 and 9 can be constructed through *criss\cross* placement of sticks, strips or lines (check ruled notebook can be found handy). Here *junctions* are counted (not intervals).



$1 \times 6 = 6$



$2 \times 6 = 12$ and so on.



Square ruled page with five horizontal lines and four vertical lines chosen.

Junctions are taken in to count and 5 time table is written:

$1 \times 5 = 5$

$2 \times 5 = 10$

$3 \times 5 = 15$

$4 \times 5 = 20$ and so on up to 9 times 5.

§ On the play ground this can be demonstrated with arrestingly ocular appeal for some children.

(3) Doubling is within the easy reach of many a child. This can be the basis for building multiplication tables.

O	O	$2 \times 1 = 2$
OO	OO	$2 \times 2 = 4$
OOO	OOO	$2 \times 3 = 6$
OOOOO	OOOOO	$2 \times 5 = 10$ 00 00 00 00 00
OOOOOO	OOOOOO	$2 \times 6 = 12$ 00 00 00 00 00 00
again for 7		
OOOOOOO	OOOOOOO	$2 \times 7 = 14$ 00 00 00 00 00 00 00
OOOOOOOO	OOOOOOOO	$2 \times 8 = 16$ 00 00 00 00 00 00 00 00
Also for 9,		
OOOOOOOOO	OOOOOOOOO	$2 \times 9 = 18$ 00 00 00 00 00 00 00 00 00

§ 3 times table can be fixed up if it is realized that a number added to 2 times the number is 3 times the number.

1×3	OOO	OO	O	$1 \times 2 + 1 = 3$	$2 \times 1 + 1 = 3$
2×3	OOO	OO	O	$2 \times 2 + 2 = 6$	$2 \times 2 + 2 = 6$
	OOO	OO	O		
3×3	OOO	OO	O	$3 \times 2 + 3 = 9$	$2 \times 3 + 3 = 9$
	OOO	OO	O		
	OOO	OO	O		
4×3	OOO	OO	O	$4 \times 2 + 4 = 12$	$2 \times 4 + 4 =$
	OOO	OO	O		12
	OOO	OO	O		
	OOO	OO	O		and so on.

This approach is liked by children with flair for manipulation.

4 times table is nothing but 2 times 2 times tables, and so it can be constructed in that way.

Now 5 times tables is nothing but combination of 2 times and 3 times table.

5×1	OOOOO	OO	OOO	$2 \times 1 + 3 \times 1 = 5.$
5×2	OOOOO	OO	OOO	$2 \times 2 + 3 \times 2 = 10.$
	OOOOO	OO	OOO	
5×3	OOOOO	OO	OOO	$3 \times 3 + 2 \times 3 = 15.$
	OOOOO	OO	OOO	
	OOOOO	OO	OOO	

6 times table and 9 times basic tables can be constructed.

6×1	O O O O O O	OO OO OO		3×2 $6 \times 1 = 6$
6×2	OO OO OO OO OO OO	OO OO OO OO OO OO		$3 \times 2 + 3 \times 2$ $= 6 \times 2 = 12.$
6×3	OOO OOO OOO OOO OOO OOO	OO OO OO OO OO OO	O O O O O O	$6 \times 2 + 6 \times 1$ $= 6 \times 3 = 18.$

7 times table can be got from 2 times table and 5 times table.

7×1	O O O O O O O	$7 \times 1 = 7.$

7×2	OO OO OO OO OO OO OO	OO OO OO OO OO OO OO	$5 \times 2 + 2 \times 2$ $= 7 \times 2 = 14.$

8 times table can be got from 4 times table by doubling.

8×1	O O O O	O O O O	$4 \times 1 + 4 \times 1$	$= 8 \times 1 = 8.$
8×2	OO OO OO OO	OO OO OO OO	$4 \times 2 + 4 \times 2$	$= 8 \times 2 = 16.$
8×3	OOO OOO OOO OOO	OOO OOO OOO OOO	$4 \times 3 + 4 \times 3$	$= 8 \times 3 = 24.$

9 times table can be got from 4 times table doubled together with ones tables.

9×1	O O O O	O O O O	O	$4 \times 1 + 4 \times 1 + 1 \times 1 =$	$9 \times 1 = 9.$

9×2	OO OO OO OO	OO OO OO OO	O O	$4\times2+4\times2+2\times1=$	$9 \times 2 = 18$.
9×3	OOO OOO OOO OOO	OOO OOO OOO OOO	O O O	$4\times3+4\times3+3\times1=$	$9 \times 3 = 27$. and so on.

Let children discover a pattern about the behaviour of digits in addition of two digits exceeding 5 and their multiplication.

$6 + 9 = 15$	$6 \times 9 = 54$
$7 + 7 = 14$	$7 \times 7 = 49$
$7 + 8 = 15$	$7 \times 8 = 56$
$7 + 9 = 16$	$7 \times 9 = 63$
$8 + 6 = 14$	$8 \times 6 = 48$
$8 + 8 = 16$	$8 \times 8 = 64$
$8 + 8 = 17$	$8 \times 9 = 72$
$9 + 9 = 18$	$9 \times 9 = 81$

DISCOVERY:

Units digits in the sums of 5 exceeding digits becomes respectively tens digits in their corresponding products.

The tables are to be remembered through familiarity and not rote learning (recall that many things are remembered through familiarity and frequency of use and not through rote learning)

Tables are not scriptures or classics to be got by heart for being remembered. Since the tables are constructed by children, there is no fear as they can be retrieved, if forgotten. There will be no imposition of writing the tables

as many times as it suits the teacher; no odd comparison with some quickly – learning children and no insults by being called names.

Tables are to be learnt and remembered in two digits upto 5 times (stage 1) and beyond 5 times upto 9.

To facilitate reinforcement and checking, (stage 2) fingers can be used in both the palms. For products of two digits exceeding 5 fingers come handy as pictured.

Below:

ADDITION

MULTIPLICATION

FIG.

ADDITION	MULTIPLICATION
two more than 5	two more than 5
$7 + 7 = 14$	$7 \times 7 = 49$
(four stretched fingers give number of units more than ten.)	(Four stretched fingers give four *tens* and product of the number of folded fingers in one hand by the number of these folded in the other hand gives *units*.)

FIG.

Two more than 5 — three more than 5

$7 + 8 = 15$

(Five stretched fingers give number of units more than ten.)

$7 \times 8 = 56$

(Five stretched fingers give five *tens* and product of the number of folded fingers in one hand by the number of these folded in the other hand gives *units*.)

Exceptional cases:

(i)

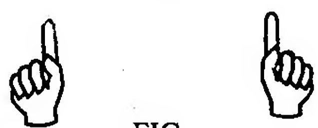


FIG.

One more than 5 — One more than 5

$6 + 6 = 12$

(Two stretched fingers give number of units more than *ten*.)

$6 \times 6 = 36$

As is observed there are 2 tens. The no. of ones is got by the product of the number of folded fingers in one hand by the number those in the other.
As 16 units give one *ten* and 6 *units*, the outcome is 3 tens and 6 units or 36.

(ii)

ADDITION — **PRODUCT**

$6 + 7 = 13$ — $6 \times 7 = 42$

FIG.

One more than 5 — Two more than 5

the explanation is obvious.

What about Tables exceeding 10?

This is an often repeated or raised query by teachers: parents and adults who learn or have learnt tables by rote learning.

Tables beyond 9 times upto twenty times are simply the combination of two tables; ten times table and table with the appropriate basic table:

They can be given as exercises in table practice:

11 times tables

$1 \times 11 = 1 \times 10 + 1 \times 1 = 11$
$2 \times 11 = 2 \times 20 + 2 \times 1 = 22$
$3 \times 11 = 3 \times 30 + 3 \times 1 = 33$
and so on

(ten times table + ones tables – taking the number of ones).

12 times table

$1 \times 12 = 10 + 1 \times 2 = 12$
$2 \times 12 = 20 + 2 \times 2 = 24$
$3 \times 12 = 30 + 3 \times 2 = 36$
and so on.

(ten times tables + twos tables – taking the number of tens and number of twos).

13 times tables

$1 \times 13 = 10 + 1 \times 3 = 13$
$2 \times 13 = 20 + 2 \times 3 = 26$
$3 \times 13 = 30 + 3 \times 3 = 39$
and so on.

(ten times tables + threes tables – taking the number of tens and number of threes).

§ Tables through progressive addition:

Progressive addition is a skill that is required in doing division. So through progressive addition and basic tables of multiplication, multiplication table of any number can be built as reinforcement as well as exercise. A few examples are given below by way of illustration

$1 \times 7 = 7$
$2 \times 7 = 14\ (7 + 7)$
$3 \times 7 = 21\ (14 + 7)$
$4 \times 7 = 28\ (21 + 7)$
$5 \times 7 = 35\ (28 + 7)$
and so on.

1×48
$2 \times 48 = 96\ (48 + 48)$
$3 \times 48 = 144\ (96 + 48)$
$4 \times 48 = 192\ (144 + 48)$
$5 \times 48 = 240\ (192 + 48)$
and so on.

It is a worthwhile exercise to shorten the progressive addition by skipping steps. Since $1 + 2 = 3$, $2 + 2 = 4$, or $1 + 3 = 4$, $2 + 3 = 5$ or $1 + 4 = 5$, $2 + 4 = 6$, or $3 + 3 = 6$, $1 + 5 = 6$ or $1 + 2 + 3 = 6$, $2 + 5 = 7$ or $3 + 4 = 7$ and so on, the tables through skipping steps is a follow up exercise in building tables requiring intelligence.

By way of illustration, multiplication table of a three digit number is given.

$1 \times 256 = 256$
$2 \times 256 = 512\ (= 256 + 256)$
$3 \times 256 = 768\ (= 512 + 256)$
$4 \times 256 = 1024\ (= 768 + 512)$
$5 \times 256 = 1280\ (= 768 + 512)$
$6 \times 256 = 1536\ (= 768 + 512 + 256)$
$7 \times 256 = 1792\ (= 1024 + 768)$
$8 \times 256 = 2048\ (= 1024 + 1024)$
$9 \times 256 = 2304\ (= 1280 + 1024)$

all the basic tables can be had in a single table as shown below (i) giving 100 facts of addition and another giving 100 facts of multiplication.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

The tables show explicitly the commutative property of addition and multiplication, additive identity 0, multiplication identity 1, annihilation property of 0!

Multiples of 9 in sequence expose the underlying pattern.

1	09
2	18
3	27
4	36
5	45
6	54
7	63
8	72
9	81
10	90

(i) Sum of the digits or the digit sum in each multiple is the same 9.

(ii) Corresponding multiple of 9 added to number of which is a multiple of 10.

Eg. 1 + 09 = 10
2 +18 = 20 and so on.

(iii) 09 and 90 are reverses of each other and 1 + 10 = 11

So are other multiples. Notice the sum of the multiples which is the same number 11.

18 and 81	2 + 9 = 11
27 and 72	3 + 8 = 11
36 and 63	4 + 7 = 11
45 and 54	5 + 6 = 11

(iv) Ten's place digits are increasing where one's (unit's) place digits are decreasing in a sequence of multiples of 9.

(v) Ten's place digit in any multiple is the *predecessor* of the number of times 9 is taken. If taken the other way one could see the successor behaviour.

(vi) One's denoting digit in any multiple and the corresponding number of times denoting digit are ten complements.

(vii) The table can be obtained through difference between corresponding tens and number of times digit.

Times digit	Difference between corresponding tens and times digit.	
1	10-1	09
2	20-2	18
3	30-3	27
4	40-4	36
5	50-5	45
6	60-6	54
7	70-7	63
8	80-8	72
9	90-9	81

(viii) Difference between a two digit multiple and its reverse is 9 times the difference of digits in either.
For e.g.: Take the difference between 81 and 18 is simply 9 times the difference between 8 and 1, which is 63.

This is deeper property.

Composite tables for addition and multiplication:

Because of the identity property of addition, identity property of multiplication, annihilative property of zero, commutative property of addition and commutative property of multiplication, number of sums to be presented get reduced to 36; the number of products to be presented also gets reduced to 36.

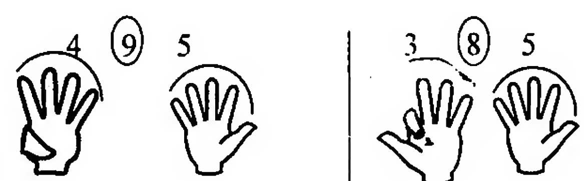
Entries are discontinued where self-commutative property prevails in each of the composite tables.

+	0	1	2	3	4	5	6	7	8	9
0		•	•	•	•	•	•	•	•	•
1		•	•	•	•	•	•	•	•	•
2		•	4	•	•	•	•	•	•	•
3		•	5	6	•	•	•	•	•	•
4		•	6	7	8	•	•	•	•	•
5		•	7	8	9	10	•	•	•	•
6		•	8	9	10	11	12	•	•	•
7		•	9	10	11	12	13	14	•	•
8		•	10	11	12	13	14	15	16	•
9		•	11	12	13	14	15	16	17	18

×	0	1	2	3	4	5	6	7	8	9
0	•	•	•	•	•	•	•	•	•	•
1	•	•	•	•	•	•	•	•	•	•
2	•	•	4	•	•	•	•	•	•	•
3	•	•	6	9	•	•	•	•	•	•
4	•	•	8	12	16	•	•	•	•	•
5	•	•	10	15	20	25	•	•	•	•
6	•	•	12	18	24	30	36	•	•	•
7	•	•	14	21	28	35	42	49	•	•
8	•	•	16	24	32	40	48	56	64	•
9	•	•	18	27	36	45	54	63	72	81

Using the composite tables, a child should be able to give the sum in addition table and the product in multiplication table at any place left blank but indicated by a dot by appropriate property.

9 times table lands itself to read the 9 multiples in their order by using fingers on both the hands.



$1 \times 9 = 9$ $2 \times 9 = 18$ and so on.

Some questions to engage the bright for exercising their intuitive powers.

(i) Use fingers on both the hands to give basic 8 times tables.

(ii) Use fingers on both the hands to give basic multiplication facts upto 5 times in each.
See page – in the book "padavaippadu" (படவாய்ப்பாடு)

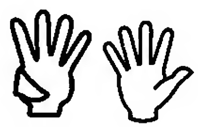
(iii) Use fingers on both the hands to do subtraction of number in respect of two one-digit numbers with their sum given and one of the digits given.

(iv) Use fingers of both the hands to do division of numbers in respect of two one-digit numbers with their product given and one of the digits given. Extend this to remainder cases as well.

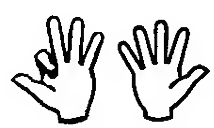
Once 9 times basic multiplication table can be presented with fingers, tables relating to multiplication involving 99, 999 can be given. See the illustration below:

$1 \times 99 = 99$ $\qquad$ $2 \times 99 = 198$

Note: folded finger is taken to think of intervening names: one 9 in the case of 99, two 9's in the case of 999, etc.



$1 \times 999 = 999$



$2 \times 999 = 2998.$

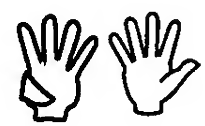
PROJECT WITH NAPIER RODS:

Napier rods are not handy or meaningful to learn basic tables without the needed preparation, which will expose their lack of naturalness.

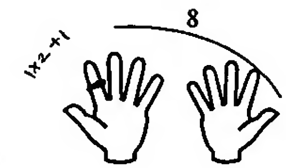
Napier rods are sometimes supplied as kits by some agencies. They are not as productive as 18 sticks through which (i) parallel placement basic addition tables can be read off (ii) through criss-cross placement basic multiplication table can be read off.

An interesting question out of curiosity surfaces at this stage. When finger folding in sequence can give the cue for getting the multiplication of table of 99, 999 etc., why not we device a strategy to get the cue of number ending in 9, starting with 19, 29, etc., Try and see.

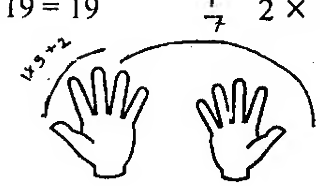
Hint:



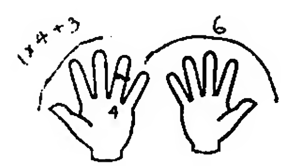
$1 \times 19 = 19$



$2 \times 19 = (1 \times 2 + 1 = 3)\ 8$ or 38



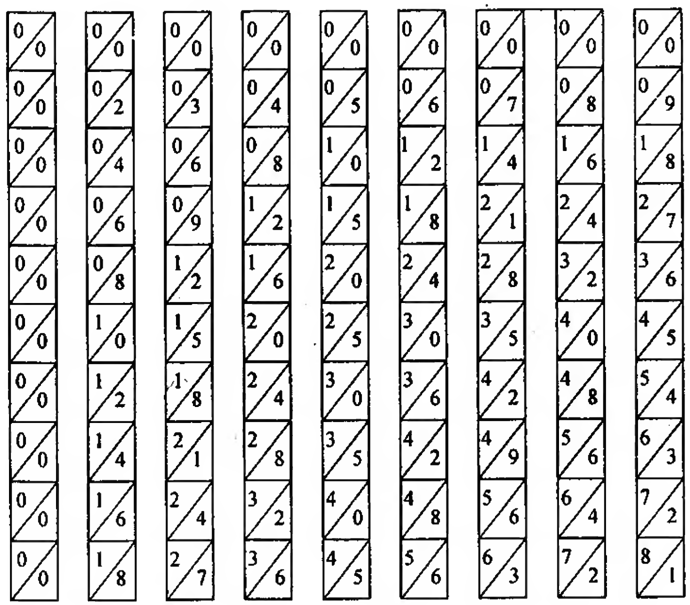
tens ones
$3 \times 19 = (1 \times 3 + 2 = 5)\ 7$ or 57



tens ones
$4 \times 19\ (1 \times 4 + 3 = 7)\ 6$ or 76 and so on.

Fix such presentation for multiplication tables of 29 to 199. This is an exciting reinforcement exercise.

Napier rods are as pictured below;

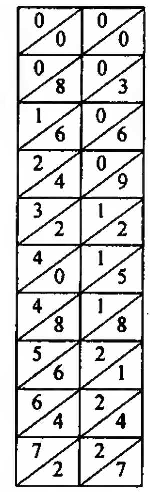


Note: If this is to be improvised a check ruled sheet is to be availed of. Strips of column of single squares with diagonal mark in each square of the strip are to be taken from the sheet to get the kit ready.

Suppose 83 times table is to be drawn up, what should be done?

Pick up the Napier rods showing 8 times and 3 times, place them side by side in alignment of sequence. The composite

rod giving 83 times gets set up and what is to be done is to read on the table and write it if need be.



$1 \times 83 = 83$
$2 \times 83 = 166$
$3 \times 83 = 249$
$4 \times 83 = 332$
$5 \times 83 = 415$
$6 \times 83 = 498$
$7 \times 83 = 581$
$8 \times 83 = 664$
$9 \times 83 = 747$

When the Napier rods are reversed 38 times table is set. Similarly, if multiplication table of a three digit number is to be got, say 769, the way is clear. Select Napier rods pertaining to 7, 6 and 9 and place them in alignment. The 769 tables shows up to read off or written.

Using the same rods, the tables of 769, 679, 697, 967 and 976 can be set and got.

[Note: This is an exercise in permutation.]

No. of digits	No. of permutation
1	1
2	2
3	6
4	24. and so on.

[Bright children discover the factorial showing up here. E.g: 3 digits giving $3 \times 2 \times 1$ and 4 digits giving $4 \times 3 \times 2 \times 2$ and so on. They need only the vocabulary and notation for communication].

A natural question arises at this juncture. What should be done if the numerals have repeated digits? The children can be seen to give the solution easily; get hold of as many sets of Napier rods as the number of times repetition occurs.

This is therefore an excellent follow up for math club projects. They can be taken up for exposition by children. Such project themes can be listed out.

Note: Once a child gets to know construction of a basic table by criss-cross method, the child knows what to do to construct any other basic table. This saves many a child from panic and insecurity that are usually witnessed when rote-learning is resorted to. This is wholesome as it ensures self esteem. What are missed in rote learning are the multiplication facts of zero which through flow technique can be got elegantly.

For example:

$4 \times 3 = 12$
$4 \times 2 = 8$
$4 \times 1 = 4$
$4 \times 0 = 0.$

(5) PROCESSES OF ADDITION AND SUBTRACTION, MULTIPLICATION AND DIVISION

1. *Addition of digits:*

If children have the experience of remembering the basic tables through constructing them, sum of two digits is easily given.

Addition of digits, exceeding two of them:

Find the sum

3 6 23(17 + 6)
alternatively
1 8 17(9 +8)
$\underline{4}$ $\underline{9}$
$\underline{8}$ $\underline{23}$

$\bullet 6^3$
$\bullet 8^7$
$\underline{9}$
$\underline{23}$

Remember only the ones, keeping the ten denoted by dot. After getting ones in the sum, the tens are given by the number of dots. This can be extended to finding sum of 4 digits, 5 digits, etc.

Noting that the digit in the sum of two single digits is always 1 facilitates rapidity of addition.

2. *Addition of two two digit numbers:*

To start with it is common to see children commit mistakes arising out of extension of experience in adding two digits.

Find the sum:

$$\begin{array}{r} 78 \\ +\,56 \\ \hline \end{array}$$

Since children know that 8 + 6 = 14 and 7 + 5 = 12, they are seen to give the sum as 1214 which is wrong. Teacher are seen frantic in making children learning carrying and giving them a lot of drill to settle down to do addition correctly by taking note of carrying figures.

If only teachers care to point out that in a place value system, there can be only one digit, drill gets reduced and children feel secure in doing addition with comfort.

$$\begin{array}{rr} 7 & 8 \\ +5 & 6 \\ \hline 12 & 14 \end{array}$$

Draw place value cubes cube in each place. Since there can be only one digit in each place, 1 of 14 represents 1 ten and it should be taken along with 7 + 5 or 12 tens giving 13 tens. Again 1 of 13 tens represents ten tens or hundred and it should be had in the hundreds place.

The teacher would ask the child to make the answer *regular*. Having more than one digit in a place value circle is *irregular*.

$$\begin{array}{rr} 7 & 8 \\ +5 & 6 \\ \hline 12 & 14 \end{array} \quad \rightarrow \quad \begin{array}{r} 78 \\ +\,56 \\ \hline 134 \\ \hline \end{array}$$

This approach when introduced has shown that children settle down to adding without mistakes and recognizing regular and irregular sum. Soon they discard this crutch and give the sum without making entries at the top as is the practice today.

Adding more than two two – digit numbers.
Find the sum

$$\begin{array}{rr} 3^8 & 8^0 \\ 4^4 & 7^2 \\ 6 & 9^5 \\ 2 & 6 \\ \hline 18 & 0 \\ \hline \end{array} \qquad \begin{array}{rr} 7^6 & 8^2 \\ 6 & 6^4 \\ 5^1 & 9^8 \\ 2 & 5 \\ 3 & 4 \\ \hline 26 & 2 \\ \hline \end{array}$$

Note: Children need to become familiar with taking $\dot{2},\dot{3}$, etc as 3,4 and so on.

The practice of having one of the numbers to be added in the mind and then adding the other number one by one is *dull and monotonous.* There is no insightful behaviour.

$$\begin{array}{r} 6 \\ +5 \\ \hline \end{array}$$ Have 5 in the mind. Show 6 fingers and say

Number-count after 5 by touching each of the 6 fingers in order.

5 • • • • • •
6 7 8 9 10 11

So the sum is 11.

Of course, having 6 in the mind and showing 5 fingers and saying number coming after 6 by touching each of the five fingers in order also gets the sum

6 • • • • •
7 8 9 10 11

Subtraction:

Present day practice that is a continuous of traditional practice is given below:

Find the difference

$$\begin{array}{r} 3\ 1\ 2 \\ \underline{1\ 7\ 8} \\ \underline{1\ 3\ 4} \end{array}$$

Can you take away 8 from 2? No, so borrowing 1 ten and take 12. Take away 8, what is left is 4. Enter it in units place.

What is left in tens place is 0. 7 cannot be taken away from 0. so look for hundreds place. There are 3 hundreds; borrowing 1 hundred or 10 tens, removing 7 from 10, 3 is obtained.

What is left in hundred place?

3-1 = 2.

Now 1 can be taken away from 2 giving 1.

This is an irksome practice. If the notion of regular and irregular form of a numeral is introduced the subtraction process becomes illuminating and meaningful.

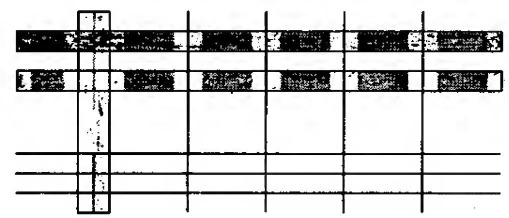
3 1 2 Do we have enough ones to take away 8 ones.
1 7 8 So we have to write the minuend in irregular form.

$$\begin{array}{ccc} 2 & 10 & 12 \\ 1 & 7 & 8 \\ 1 & 3 & 4 \end{array}$$

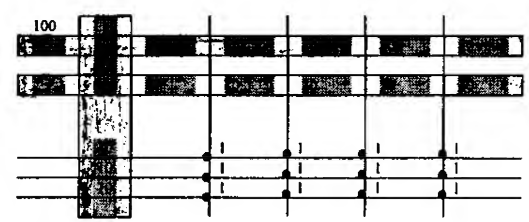
This approach has been observed to reduce drill and make the child get the difference with confidence and ease.

Multiplication

It is continuity of criss-cross technique for getting basic multiplication tables. Instead of using only strips representing ones, strips standing for tens are used.



What does this represent? 23 × 14. Making the values at the junction, the situation is as follows.



Recording the result. Multiplication know how is obtained

23 × 14	
200	20 × 10
110	20 × 4 + 3 × 10
12	3 × 4
322	

23	
× 14	
12	4 ones × 3 ones (collecting ones)
80	4 ones × 2 tens + 1 ten × 3 ones
+ 30	(collecting tens)
200	1 ten × 2 tens (collecting hundreds)
322	

This experience boils down to collecting ones, tens, hundred and so on.

One should know *ones* by *ones* give *ones*, *ones* by *ten* and *tens* by *ones* give *tens*, *ones* by *hundred*, *tens* by *tens* and *hundreds* by *ones* give *hundreds*.

To start with multiplication of multidigit numbers by a single digit is taken up (i) 'criss-cross type' and (ii) carrying type as illustrated below:

	(100)	(10)	1				
(i)	1	3	2	× 2	= 264		
	h	t	n				
	(100)	(10)	1		h	t	u
(ii)	1	3	2	× 8	=	1	6
	h	t	u			2	4
						8	
					1	0 5	6

if irregular form is preferred the product can be shown as

h t u

8 24 16

=1056

Note: There is need to know the table of place value which is not sufficiently emphasized:

10 *ones* make 1 *ten*
10 *tens* make 1 *hundred*
10 *hundreds* make 1 *thousand* and so on.

Next: Multi-digit number by a two-digit number is taken up involving (1) digits each less than 5, (2) digits each greater than 5, and (3) mixed, in that some digits are more and less than 5.

(1)

3124 × 12	
6248	3124 × 2 ones
3124	3124 × 1 ten
37488	

(2)

4106 × 82		
8212	8 2 0 12	4106 × 2 ones
3248	32 8 0 48	4106 × 8 tens
40692		

(3)

6759 × 48	48 56 40 72	
54072	48 56 40 72	ones 6759 × 8 ones
27036	24 28 20 36	tens 6759 × 4 tens

2 12 4 3 13 2

3 2 4 4 3 2

Advanced stage:

Collecting all the possible ones, tens, hundreds in the product helps one to give the product without introductory steps:

e.g.
$$\begin{array}{r} 21\ 58 \\ \times\ 34\ 26 \end{array}$$

(hundreds × thousands) (3 × 2) (ones)

$$\begin{array}{rrrrr} & 2 & 1 & 5 & 8 \\ \times & 3 & 4 & 2 & 6 \end{array}$$

(ones × ones) (6 × 8) (ones)

6 11 23 58 48 46 48

(hundreds × thousand + (thousands × hundred (4 × 2 + 3 × 1) (tens)

(ones × thousands) + (tens × hundreds) + (hundreds × tens) + (thousands × ones) (6×2+2×1+4×5+3×8) (thousands)

(ones × tens) + (tens × ones) (6 × 5 + 2 × 8) (tens)

(tens × thousands) + (hundreds × hundreds) + (thousands × tens) (2 × 2 + 4 × 1 + 3 × 5) (hundreds)

(ones × hundreds) + (tens × tens) + (hundreds × ones) (6 × 1 + 5 × 2 + 4 × 8) (hundreds)

Sometimes multiplying two powers of ten are done clumsily by multiplying by zeroes of each place. This needs to be avoided.

For example; how to find. 200000 × 3000
what does this mean 2 lakhs × 3 thousands
= 6 thousand lakh. = 600000000

Note: Before division process is taken, there should be exposure of children finding outcomes when the result of product addition to a number is formed.
This will create readiness to check division later requiring Dividend = divisor × quotient + remainder.

Division process

Stage 1:

Division of a multi-digit number by a single-digit number.

$4856 \div 7$

```
    0693
 7) 4856
    42
     65
     63
      26
      21
       5
```

Quotient is 693, remainder 5

Checking $693 \times 7 + 5 = 4856$

Four thousands cannot able distributed equally in 7 places as thousands. So four thousands are changed to hundreds giving 40 hundreds. Along with 8 hundreds there are 48 hundreds.

48 hundreds can be distributed in 7 places equally, each place to have 6 (as $7 \times 6 = 42$). 6 hundreds changed to tens give 60 tens. Together with 5 tens, there are 65 tens. 65 tens can be distributed equally in 7 places, each place getting 9 tens. Remaining 2 tens, 2tens changed to ones and 6 ones are added giving 26 ones. 26 ones distributed equally in 7 places, each place getting 3 ones, leaving finally the remainder of 5 ones.

Stage 2:

Division of a multi-digit number by a two-digit number.

Illustration:

$2735 \div 48$

Keep the 48 multiples
Table ready using progressive addition.

1	48
2	96
3	144 (96+48)
4	192 (144+48)
5	240 (192+48)
6	288 (240+48)
7	336 (288+48)
8	384 (336+48)
9	432 (384+48)

```
       56
 48) 2735
     240
      335
      288
       47
```

Note: children can be taught mechanically but their appreciation of what is being done will be impaired. One way of checking is to give the place value of partial dividends or partial products.

What is the place value of 2 in 240?
It is not 2 hundreds but 2 thousands as 240 means 240 tens.

The ultimate remainder needs to be less than the divisor. In case it is more, there is no need for panic. Division is to be continued. That is all

```
   59
   1
   58
48) 2835
   240
    435
    384
     51
     48
      3
```

It is often noticed that mistakes occur when the partial quotient gives partial product.

```
   55
3) 1515
   15
    15
    15
     0
```

Children are not instructed that every place should be accounted for

```
  0505
3) 1515        the quotient is 505 and not
55.
 0
 15
 15
   1
   0
    15
    15

    0
```


Stage 3:

Checking digit by digit of the dividend to fix the partial dividend

```
48) 2735          2 < 48
                 27 < 48
      56        273 > 48
48) 2735
    240
    335
     288
      47
```

Hence 273 has to be chosen.
48 is nearing 50.
5 goes into 27, 5 times.
Check the 'carrying' figure.
It works out,
5 goes in to 33,
6 times so take 6 times 48. If, it is not suitable, reduce the figure.

Note: Since calculators, have come into use, there is no need to spend time exhaustively in learning division in middle decades.

(6) NUMERATION AND PLACE VALUE SYSTEM

The attention paid to learning of numeration and place value system is seen to be inadequate. It is instrumental for poor understanding and conceptual hurdles. Children in general need experience in handing concrete objects, both continuous and discrete in developing the ability and skill to handle with confidence and care place value numeration.

The first stage till ten objects are counted and ten should not be introduced to be written as 10 unless it is understood that 1 means 1 ten and 0 means '0' ones. This is possible only after bundling of sticks is done.

Once children know to count ten objects, let children make bundles of ten sticks each using rubber band. This is liked by children.

Ask them to give as many tens as asked for and as many ones specified.

To start with let children learn that one ten is ten, two tens are twenty, three tens are thirty and so on. If children prefer saying onety one, twoty two, it can be allowed for certain time, till they know that it is not the normal practice and they will discontinue.

Let them tabulate their outcomes and let them lead the number of sticks available each time.

		t	u	
2 ten	3 ones	2	3	Twenty three
5 tens	6 ones	5	6	Fifty six
8 tens		8		Eighty (what about ones? zero)
9 tens	7 ones	9	7	Ninety seven
	5 ones		5	Five (what about tens? zero)

Once this introduction is made let children make numbers 1 to 99.

1	6	11	:	90	95
2	7	12	:	91	96
3	8	:	:	92	97
4	9	:	:	93	98
5	10	:	:	94	99

99 and one more is hundred. Let children get introduced to hundred in this way. There are 9 bundles of ten sticks each and loose sticks 9 in number. 9 loose sticks and one more stick taken along give ten sticks; on the whole, ten bundles of ten-stick bundles, each such bundle tied with bigger rubber bands which is cumbersome, be given squares (cut out from squares ruled sheets) each showing ten rows of ten small squares or hundred.

Strips showing ten small squares and small squares showing ones have also to be used.

As before let children be instructed to tabulate their responses requiring selection of counts in *hundreds, tens* and *ones*.

				H	T	U	
3 hundreds	5 tens	and	7 ones	3	5	7	Three hundred and fifty seven.
4 hundreds			6 ones	4		6	Four hundred and six
8 hundreds	2 tens			8	2		Eight hundred and twenty
9 hundreds				9			Nine hundred

Let children insert zero where there is absence of count, showing the advantage obtained though dropping of place headings.

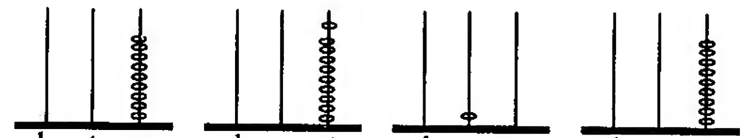
3	5	7
4	0	6
8	2	0
9	0	0

After reaching the stage of recognising numeration up to 999 with meaning of place value of each digit in it, let children be introduced to abacus with removable rods or spikes and insertable beads in boxes of ten.

Elicit knowledge about the emerging table of place value in two ways:

10 ones make I ten	1 ten gives 10 ones
10 tens make 1 hundred	I hundred gives 10 tens.

Let children use abacus with three rods inserted and three boxes of ten beads each (readily inser able).

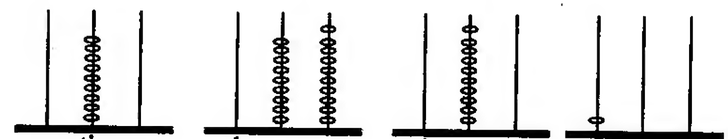


h t u
9

h u t
9 and one more

h t u
1 ten

h t u
9 tens after showing 11 to 90

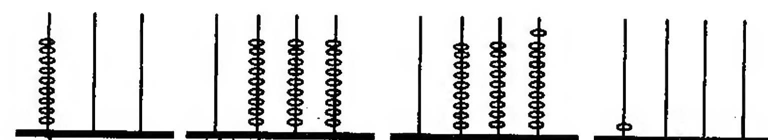


continue showing 91 to 99

h t u
Put one more, you get 9 tens and 10 ones.

h t u
10 tens are got

h t u
one hundred is got



h t u
Continue showing hundred till 9 hundreds are got

(th) h t u
Continue putting the beads to show 901 to 999 and insert one more rod to the left.

(th) h t u
Insert one more bead in the units column and make the cnanges needed.

(th) h t u
The newly inserted rod gets one bead. Give the name one thousand.

Note: It is necessary that beads should be inserted one by one and every number accounted for to save time.

Continue the place value table.

10 hundreds make 1 thousand 1 thousand gives 10 hundreds.

Once this pattern of place value is comprehended, even the semi concrete or semi abstract abacus representation can be discontinued. Instead, the place value table can be extended.

10 thousands make 1 ten thousand	1 ten thousand gives 10 thousands
10 ten thousands make 1 hundred thousand or 1 lakh	1 hundred thousand gives 10 ten thousand.
10 hundred thousand or 10 lakhs make 1 million or 1 ten lakh	1 million or 1 ten lakh gives 10 hundred thousands or 10 lakhs.
10 millions or 10 ten lakhs makes 1 ten million or 1 crore	1 ten million or 1 crore gives 10 million or 10 ten lakhs.

Naming place value can stop at this stage though without name the numeration can be continued.

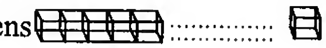
Naming is had in two systems, Indian and International systems.

In Indian system, the first three digits get marked and then marking off sequence is by two digits at a time.

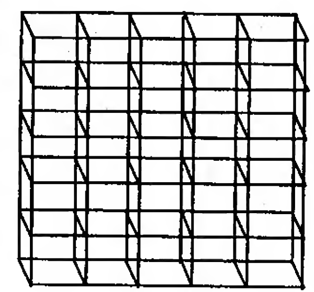
But in International system, the marking off sequence is always with 3 digits at a time.

Place value naming in International aspects can be visualized and appreciated.

Have objects made of cubes.

Individual cubes are called *units*  . Ten of them fixed in a row is called a *long* to show tens  . Ten such longs placed lengthwise and fixed is called a *flat* to show hundreds. Ten such flats placed one on the other form a *block*, to show thousand.

This can be continued by starting with block as higher unit standing for thousand. Ten of them fixed in a row is called a bigger long or higher long to show ten thousand. Ten such higher long placed side by side and fixed together to form a bigger or higher flat to show hundred thousand. Ten such higher flats piled one over the other to form a bigger block or higher block to show thousand or million.



This is not to be done by children. It is enough if children are shown these models made out of balsa wood.

This numeration scheme can be presented in the form of tables, which textbooks generally have:

INDIAN SYSTEM

1 hundred crore		1 crore		1 lakh		1 thousands		
Hundred crore	Ten crore	Hundred lakhs	Ten lakhs	Hundred thousands	Ten thousand	Hundred	Ten	Ones

INTERNATIONAL SYSTEM

Thousand thousand million (American)			Thousand million (or) billion (American)			Million			Thousand		
Thousand billion	Hundred billion	Ten billion	Thousand million	Hundred million	Ten million	Thousand thousand	Hundred thousand	Thousand	Hundred	Tens	ones

Place value or P.V. table need to be remembered like basic addition and basic multiplication tables and there should be frequent warming up exercise in place value table too.

EXPANDED NOTATION.

Importance is given in the curriculam as well in the textbooks giving expanded notation of multi digit numeral.

Expanded notation needs to be present in various ways in all classes 1 to 8 till standard scientific notation is reached. To facilitate better understanding, partially expanded notation and like and unlike denominator numerals need to

be introduced. They are now not thought of by curriculum makers and textbook makers and more so the classroom teacher.

The following illustration pin this innovation:

25168 = 2 ten thousand 5 thousand 1 hundred 6 tens 8 ones
= 25 thousands 1 hundred 68.

[*Note* : The practice of leading this numeral as 25 thousand 1 hundred and 68 is to be discouraged as it is a hang over of the British tradition which gets changed in Nuffield Foundation guide lines and accepted in NCERT books.]

= 2 tens thousands 5168 ones.
25 thousands 168 ones
251 hundreds 68 ones.
2516 tens 8 ones.

[*Note*: 5 thousand and 8 hundred are unlike denomination. They need not be changed to ones to have them in like-denomination. Instead lower denomination given can be noted and used in giving this in like denomination. Here it is 50 hundreds 8 hundreds.]

$$
\begin{aligned}
25168 &= 20000+5000+100+60+8 \\
&= 2 \times 10000 + 5 \times 1000 + 1 \times 100 + 6 \times 10 + 8 \\
&= 2 \times 10^4 + 5 \times 10^3 + 1 \times 10^2 + 6 \times 10^1 + 8 \\
&= 2 \times 10^4 + 5 \times 10^3 + 1 \times 10^2 + 6 \times 10^1 + 8 + 10^0
\end{aligned}
$$

[*Note:* when this is reversed, children's attention to place value gets reverted]

Few examples are given below:

- $30000 + 60 = 30000 + 000 + 000 + 60 + 0 = 30060$
- $3 \times 10^6 + 5 \times 10^4 + 7 \times 10^3 + 4$
  $= 3 \times 10^6 + 0 \times 10^5 + 5 \times 10^4 + 7 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 4$
  $= 3057004$
- $800507 = 8 \times 10^5 + 5 \times 10^2 + 7$
- $6 \times 10^5 + 4 \times 10 = 600040$

Children should get used to treating 10^5 as 1 followed by 5 zeroes, 10 as 10^1, 1 as 10^0, before getting introduced to index laws.

Also children have become familiar with index laws of multiplication invoving higher units.

$1000 \times 100 = 100000$: $10^3 \times 10^2 = 10^5$ (10^{3+2})
$10000 \times 10000 = 100000000$; $10^4 \times 10^4 = 10^8$ (10^{4+4}) and so on.

This will give the children the skill to check the number of places in product of 2 numbers. See the illustration.

$3804 \times 512 = 3 \times (10^3 + \ldots) . (5 \times 10^2 \ldots)$
$= (15 \times 10^5 \ldots)$

* This can be extended to division:

$100000 \div 1000 = 100$	$10^5 \div 10^3 = 10^2$ (10^{5-3})
$10000000 \div 100 = 100000$	$10^7 \div 10^2 = 10^5$ (10^{7-2})
$1000 \div 100 = 10$	$10^4 \div 10^2 = 10^2$ (10^{4-2})
$100 \div 100 = 1$	$10^2 \div 10^2 = 10^0$ (10^{2-2})

An exciting event

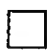
Show 312; change 321 to 132, and so on. Check arrangements of place value.

Use finger and rings. Five digit number can be presented and named. Of course, it should be made known that no finger can have more than 9 rings on the second finger of higher place value.

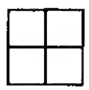
(7) SQUARES, CUBES, TRIANGULAR NUMBERS AND POWER

Recognition of a square, a cube and a triangular number arms a child with keenness in handling them and finding their inter-relationship.

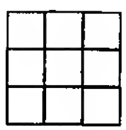
Square



$1 \times 1 = 1$



$2 \times 2 = 4$



$3 \times 3 = 9$

.
.
.
.

$9 \times 9 = 81$

Square numbers up to 100 are 1,4,9,16,25, 36,49,64,81.

Note: This is not true of these numbers alone but true of all numbers, which are countless. How can you say that? Since any number ends in one of these 9 digits, its square should have to end in only the digit 1 to 9 with 2,3 & 7 & 8 excluded. This is called *proof by exhaustion.*

First discovery:

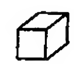
No perfect square number up to 100 (and extended later by reasoning about the last digit of any number and the square of that number ending with square of any of these 9 digits) ends in 2,3,7 and 8.

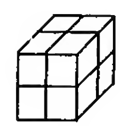
Since $10 \times 10 = 100$
$100 \times 100 = 10000$

It is also discovered that no higher units can have odd numbers of zeroes if it is to be a square number.

Cubes:

Forming cubes start with unit cube 1^3. Build 2^3 using the unit cube as shown (roughly). Continue the exercise with 3^3 and so on as shown in the table formation.





	Front	Annexing	Top
1^3			
2^3	1	2	2×2
3^3	2×2	3×2	3×3
4^3	3×3	4×3	4×4
5^3	4×4	5×4	5×5
.			
.			
6^3	5×5	6×5	6×6

As in the case of square numbers, children can articulate their discovery.

A cube number ends in all digits unlike square numbers.

Square number ending digits

1 4 9 6 5 6 9 4 1 00
1 $\underline{8}$ $\underline{7}$ 4 5 6 $\underline{3}$ $\underline{2}$ 9 000

Some more discoveries:

Cube numbers are more exciting.
Cubes of numbers have the same units place digit if they end in 1,4,5,6, and 9.

When numbers end in 8 and 7, their cubes end in 2 and 3 and vice versa, digits 2,3,7,8 which vanishes in square, make their appearance. Moreover 2 and 8, 3 and 7 are ten complements. There cannot be any child who is not wonder struck by this number behaviour.

This behaviour enables to set cultural show during school anniversary or prodigy feat in math clubs.

Given a cube number up to 1000, the number of which it is cube is at once given providing an opportunity for the performing child to get applause and feel pleasure.

(Note: This can be extended to cube beyond thousand in middle and high level).

In the case of squares

Digits	ending digit in square
1 & 9	1
2 & 8	4
3 & 7	9
4 & 6	4
5	5

no child exposed to discovery stance in learning climate fails to see that the digits having the same ending or units place digits in the square are complements of 10.

TRIANGULAR NUMBERS:

It is better to introduce oblong numbers before thinking of triangular numbers.

If the number of rows and number in each row are successor number, the array gives an oblong number such as illustrated below:

O O O	O O O O	O O O O O
O O O	O O O O	O O O O O
	O O O O	O O O O O
		O O O O O

If they are formed with bottle tops, half of them in each oblong array can be reversed to exhibit triangular numbers and their formation.

● ● O ● ● ● O ● ● ● ● O
● O O ● ● O O ● ● ● O O
　　　● O O O ● ● O O O
　　　　　　　　● O O O O

$2 \times 3 = 2\,(1+2)$
$3 \times 4 = 2\,(1+2+3)$
$4 \times 5 = 2\,(1+2+3+4)$

$1+2 = \frac{1}{2}\,(2 \times 3)$

$1+2+3 = \frac{1}{2}\,(3 \times 4)$

$1+2+3+4 = \frac{1}{2}\,(4 \times 5)$ and so on

Note:

Given $1 = \frac{1}{2}\,(1 \times 2)$
the initial position gets cleared.

What happens next? Without the visuals, that is to say at the take off stage children are able to state with confidence

that $(1+2+3+ \ldots +100) = \frac{1}{2} \times 100 \times 101$
$= 5050$

♦ Cultivation of mathematical eye is an avenue for curiosity and investments in one's ability to make discoveries unprodded.

Consider square number visuals

O O O O O O O O O
O O O O O O O O O
　　O O O O O O O
　　　　　 O O O O

Sights viewed through mathematical eye are varied

O O ● ● ● O O O O
● O O O ● ● ● ● O
　　● O ● O O ● O
　　　　　 ● O ● O and so on.

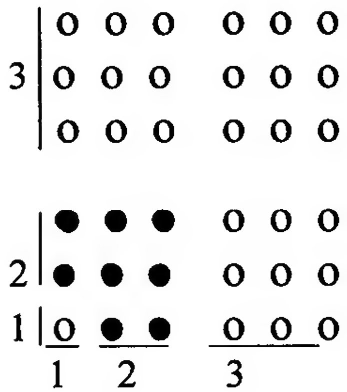
$1 + 3 = 4$　　$1 + 3 + 5 = 9$　　$1 + 3 + 5 + 7 = 16$

The start should have been. ● 1 (1×1 or 1 square number) Sum of consecutive odd numbers starting from 1 is a square number – a beautiful property indeed to be enjoyed in the primary class itself.

Further sight viewed through mathematical eye

O O O
O O O　　$1 + 8 = 9$ or $(1+2)^2$
● O O　　1 and 8 are cube numbers.

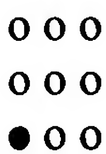
Can 1+8+27 form a square? Yes it is. It is 36 or 6^2. Can this be seen as a visual? Try and see.

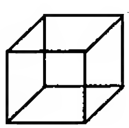


A beautiful visual is formed and it is seen to be

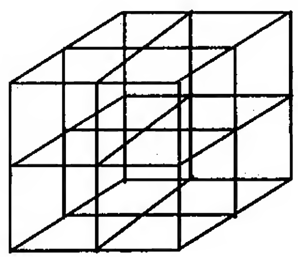
$$1+8+27 = (1+2+3)^2$$

To get arrestingly appealing effect, use of cubes will be welcome.





1(1 × 1 × 1)

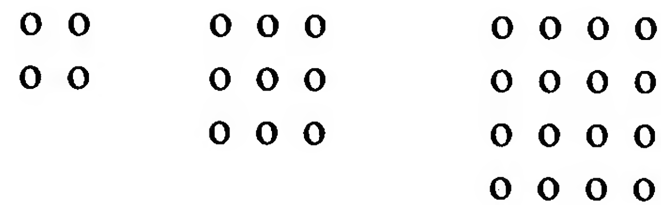


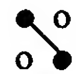
8 (2 × 2 × 2)

And so on giving the number behaviour.

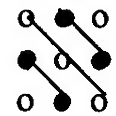
$$1^3 + 2^3 + 3^3 + \ldots\ldots = (1 + 2 + 3 + \ldots\ldots)^2$$

- One more sight viewed through mathematical eye can be taken up.

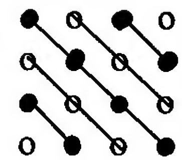




$1+2+1 = 2^2$



$1+2+3+2+1 = 3^2$



$1+2+3+4+3+2+1 = 4^2$

and so on.

(The first one to be seen as $1=1^2$)

Incidentally, each visual is seen to consist of two consecutive triangular numbers.

$\frac{1+2}{2^{nd}}$		$\frac{+1}{1^{st}}$	$\frac{1+2+3}{3^{rd}}$		$\frac{+2+1}{2^{nd}}$	$\frac{1+2+3+4}{4^{th}}$		$\frac{+3+2+1}{3^{rd}}$
triangular number	+	triangular number	triangular number	+	triangular number	triangular number	+	triangular number

The emerging pattern is

$2^2 = 2^{nd}$ triangular number + 1^{st} triangular number.

$3^2 = 3^{rd}$ triangular number + 2^{nd} triangular number.

$4^2 = 4^{th}$ triangular number + 3^{rd} triangular number

These can be presented visually as well and seen.

● O   ● O O   ● O O O
● ●   ● ● O   ● ● O O and so on.
       ● ● ●   ● ● ● O
               ● ● ● ●

These patterns of number behaviour can be presented on the stage as performance with appropriate costumes and movements with background music or on the playground with suitable head wears.

Take the sequence of *whole numbers* 0,1,2,3,4,5 and so on. Sum of any two whole numbers is again a whole number. It is so always. This goes by the closure property of whole numbers. Now take the sequence of squares of natural numbers.

1 4 9 16 25 36 49 64 81 100 etc.
is the sum of two squares a square? One would see that it is not always a square number.

E.g.

1+4 = 5 not a square number
1+9 = 10 not a square number
4+9 = 13 not a square number
9+16 = 25 is a square number

Looking at another way 5 is expressed as a sum of two squares; so is 10 and so on.

One has to search for such cases. It is but natural to find out the first instance when a number can be expressed as a sum of two squares in two different table of square and fill up looking for this property. In other words, start with a few squares in the row and in the column of the table. The table can be continued only when needed.

+	1	4	9	16	25	36	49	64	81	100
1	2	5	10	17	26	37				
4	5	8	13	20	29	40				
9	10	13	18	25	34	45				
16	17	20	25	32	41	52				
25	26	29	34	41	50	61				
36										
49										
64										
81										
100										

+	1	4	9	16	25	36	49
1	2	5	10	17	26	37	50
4	5	8	13	20	29	40	53
9	10	13	18	25	34	45	58
16	17	20	25	32	41	52	65
25	26	29	34	41	<u>50</u>	61	74
36	37	40	45	52	61	62	85
49	<u>50</u>	53	58	65	74	85	98

The quest is over with finding that 50 is the first such number that can be expressed as a sum of two squares in two different ways:

$$50 = 7^2 + 1^2 = 5^2 + 5^2$$

§ One's curiosity is roused at this juncture to find out if such things can happen with cubes:

+	1	8	27	64	125	216	324	729	1000
1	2	9	28	65	126	217	325	730	1001
8	9	16	35	72	133	224	332	737	1008
27	28	35	54	91	152	243	351	758	1029
64	65	72	91	128	189	280	388	793	1064
125	126	133	152	169	250	341	449	854	1125
216	217	224	243	280	341	512	540	945	1116
324	325	332	351	388	449	540	642	1053	1324
729	730	737	756	793	854	945	1053	1458	1729
1000	1001	1008	1027	1064	1125	1216	1324	1729	2000

Table needs to be expanded further

1331	1332	1339	1358	1395	1456	1547	1655	2060	2331
1728	1729								
2197									

The question is fruitful. The first number that can be expressed as a cube in two different ways is 1729, which is $12^3 + 1^3$ or $10^3 + 9^3$.

[*Note:* This number is famous in the life story of Ramanujan. When he was in the sanatorium, his mentor Prof. Hardy visited him. His car had the number 1729. To draw Ramanujan out from morose state of mind caused by illness, he started telling Ramanujan that his car number is 1729 and it happens to be a dull number in that it is odd and its factors are also odd $7 \times 13 \times 19$.

At once Ramanujan warmed up and observed, 'No, Prof Hardy, it is the smallest of the numbers that can be given as a sum of two cubes in two different ways.']

This can be given as vacation projects with option to take it up if interested.

Experiences such as these whets one's curiosity to know more and about fascinating patterns of number behaviours.

[Note: It was Euler who gave the smallest number which can be expressed as a sum of two fourth power of natural numbers in two different ways. Unless one is obsessive, it is better to know the case and stop.

$$2338^4 + 3351^4 = 3494^4 + 1623^4$$
$$= 155, 974, 778, 565, 937]$$

(8) WHOLE AND PART AND FRACTION

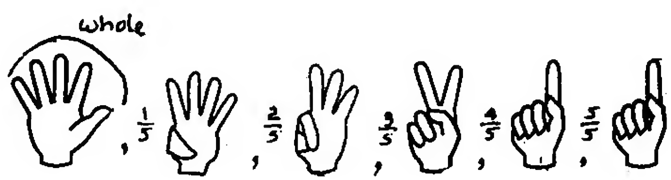
Preconceived notions of whole and part vitiate understanding the notion of fractional number from its initial stage meaning a whole number divided by a natural number.

I have often asked the teacher, present in numerous workshops I have conducted, 'Are you a whole? ' This teacher invariably replies, 'I am a whole'. When asked if he/she is not a *part* of his / her family, he / she smiles and says,. 'of course'. Then it dawns on the minds of all the workshop participants that one can be a whole or a part depending on the consideration.

Whole and part concept needs to be had in continuous as well as discreet situation.

Taking one's five fingers in one hand to make a whole, what part of the whole is each finger? It is one out of five or one fifth written as 1/5.

Two fingers form 2 out of 5 or two fifths written as 2/5 and so on till five out of five or five fifths written as 5/5, is obtained whole.



If four fingers are taken together to form a whole naming the fingers in sequence of one, two etc. as parts of the whole considered, gives 1/4, 2/4, 3/4, 4/4 or 1. What about the fingers outside or beyond the whole? It is 1/4 more than the whole (4/4) or 5/4.

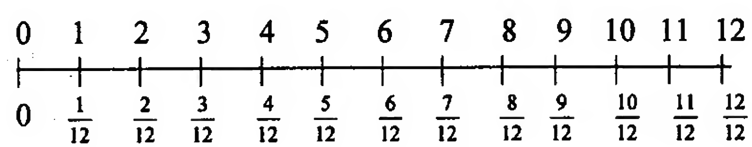
This stage will have to be taken up after children have had elementary notation of half, quarter, a third, etc. to start with in earlier classes (half of a rectangular sheet, quarter of a circular sheet, etc).

- A foot ruler can next be taken up to deepen the notions of whole and part.

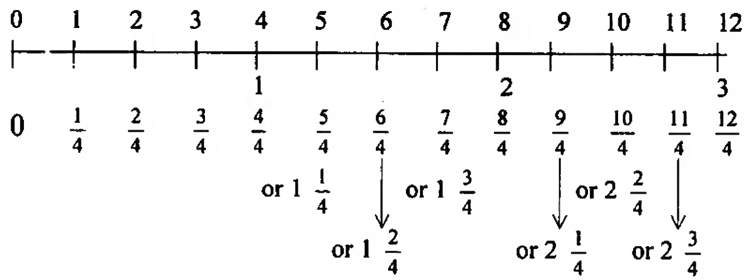
I have again asked the teachers in the workshop, each having a foot rule, 'Is the foot rule to be taken as a whole?' the answer is expectedly 'yes'. Ask the participants to see where mark 1 is given. Confusion can be seen to prevail. One inch along one edge or one cm along the opposite edge is taken as *a whole* and there are 12 wholes along the edge showing inches or 30 wholes along the edge showing centimeter. If the entire foot rule is taken as a whole, then what does 1 mark means? It is one out of twelve or one twelfth or 1/12 followed by 2/12, 3/12 etc, till 12/12 or 1 is reached. On the other edge it starts with 1/30 followed by 2/30, 3/30, etc. till 30/30 or 1 is reached.

Very few teachers or textbook writers take care to bring this illumination in learning of fraction, as there is no explicit mention in the syllabus or curriculum.

This experience gets strengthened when whole is changed and parts named.

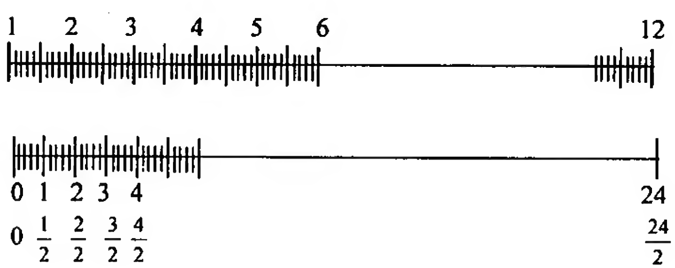


Suppose 1 is taken at the division marked 4, how do the parts get named?



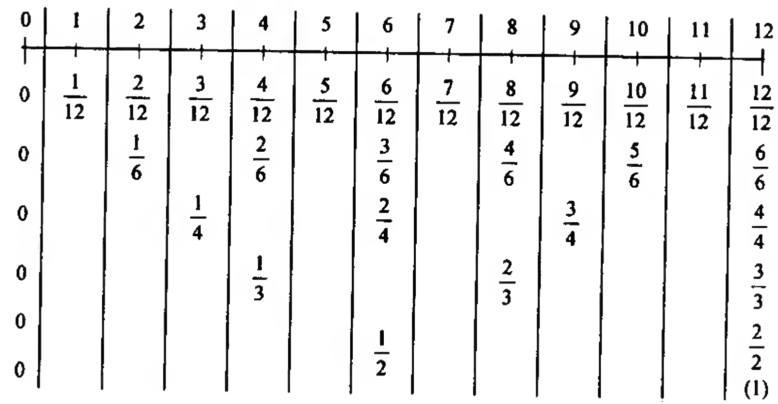
Allow children to change the whole and name the parts and settle down to the concept that *whole and part are relative.*

Some precocious children are seen to fix 1 midway between 1 and 1 on the edge of the foot rule showing inches or on the edge showing centimeter, as there are markings available and take up the exercise of naming the other parts.



Let children take a check ruled sheet or ruled sheet with marginal line and go about having the naming spree of whole and part.

§ Equivalent fractions:



To impress on the teacher that more than one name, there can be for the same thing. 'I call a teacher and put the question,' How many names do you have? The reacher is seen to respond rather disturbingly, 'I have only one name, "Rani". I asked some more details about her family. She continues saying that her son was Kumar, husband Muthu, brother Vivek and so on.' I put to the class as a whole, 'Kumar's mother'.

Rani is seen to get up. I remark, so you have another name. "Kumar's mother". Now I call her as Mr. Muthu's wife. Again Rani identifies herself. Now she has a third name, Mr. Muthu's wife. Then I ask for Mr. Vivek's sister. She is seen to answer again. The entire class realizes that every relationship gives a name to a person.

In the same analogy one could see that ½ of a whole is the same as 2/4 of the whole.

3/6 of the whole, 6/12 of the whole.

Children are then asked to write parts with different names.

They are seen to write.

$$\frac{2}{12}=\frac{1}{6}$$
$$\frac{3}{12}=\frac{1}{4}$$
$$\frac{4}{12}=\frac{2}{6}=\frac{1}{3}$$
$$\frac{6}{12}=\frac{3}{6}=\frac{2}{4}=\frac{1}{2}$$
$$\frac{8}{12}=\frac{4}{6}=\frac{2}{4}=\frac{1}{2}$$
$$\frac{9}{12}=\frac{3}{4}$$
$$\frac{10}{12}=\frac{3}{4}$$
$$\frac{12}{12}=\frac{6}{6}=\frac{3}{3}=\frac{2}{2}$$

Ask children if they can give these equalities without the diagram.

A few children discover the property and say.

When the numerator and denominator of a fraction are multiplied by the same natural number, the resulting fraction is a new name.

Also, when the numerator and the denominator of a fraction are divided by their common factor, the resulting fraction is another name.

In reinforce this understanding discrete situation is presented for children to do this naming and finding different fractions naming the same entity.

1
0000000000000000

16 objects taken as a whole.

00000000	00000000
1/2	2/2

Two group of 8 objects in each.

0000	0000	0000	0000
1/4	2/4	3/4	4/4

Four group of 4 objects in each.

00	00	00	00	00	00	00	00
1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8

Eight groups of 2 objects in each.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{5}{16}$	$\frac{6}{16}$	$\frac{7}{16}$	$\frac{8}{16}$	$\frac{9}{16}$	$\frac{10}{16}$	$\frac{11}{16}$	$\frac{12}{16}$	$\frac{13}{16}$	$\frac{14}{16}$	$\frac{15}{16}$	$\frac{16}{16}$

16 objects taken separately.

Looking for fraction naming the same number of objects in different group, children readily write.

$$\frac{1}{2}=\frac{2}{4}=\frac{4}{8}=\frac{8}{16}$$
$$\frac{1}{4}=\frac{2}{8}=\frac{4}{16}$$
$$\frac{3}{8}=\frac{6}{16}$$
$$\frac{4}{8}=\frac{8}{16}$$

$$\frac{5}{8}=\frac{10}{16}$$
$$\frac{3}{4}=\frac{6}{8}=\frac{12}{16}$$
$$\frac{7}{8}=\frac{14}{16}$$
$$\frac{2}{2}=\frac{4}{4}=\frac{8}{8}=\frac{16}{16}$$

The same property to get fractions naming the same part in different ways is spelled out again.

Combining both the experiences, children are in a position to understand the concept of equivalent fractions. With the property that when the numerators and the denominators are multiplied by the same counting number, equivalent fractions are obtained.

The number of equivalent fractions for a given fraction is countless as multiplying numbers are countless.

As a fitting finale to this experience and the accompanying conceptualization children continue the equivalent fraction as shown below:

$$\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=---=---=--- \text{ and so on.}$$
$$\frac{1}{3}=\frac{2}{6}=\frac{3}{9}=---=---=---=--- \text{and so on.}$$

§ ***Kinds of fractions:***

Given two fractions, how to find which is greater or smaller and which is equal is the question.

There is the need for vocabulary as spelled out below:

Fractions in a row naming difference numbers of parts into which a whole is divided are *like fractions* revising at the same time proper and improper fractions.

$\underbrace{\frac{1}{4},\frac{2}{4},\frac{3}{4}}_{\text{Proper}},\ \underbrace{\frac{4}{4},\frac{5}{4}}_{\text{improper}}$ and so on are like fractions.

Proper improper

Fractions with 1 in the numerator are unit fractions.

Eg. $\frac{1}{2},\frac{1}{3},\frac{1}{4}$ and so on.

Fractions with different denominators are *unlike fractions*

$\underbrace{\frac{2}{3}, \frac{3}{4}, \frac{4}{7}, \frac{5}{8}}_{\text{Proper}}$ $\underbrace{\frac{9}{9}, \frac{11}{10}, \frac{13}{12}}_{\text{improper}}$ and unlike fractions.

Comparison of unit fractions:

Looking back at the diagrams showing continuous situation and discrete situation and discrete situation and discrete situation, and noting inequalities, two observations are immediate:

$\frac{2}{3} > \frac{2}{5} > \frac{2}{8}$, etc when the numerator is the same, the fraction with smaller denominator is greater. In other words, of two unlike fractions with the same numerator, the one with smallest denominator is greater.

$\frac{4}{6} > \frac{2}{6} > \frac{1}{6}$, etc. When the denominator is the same the fraction with greater numerator is greater. In other words, of two like fractions, the one with greater denominator is greater.

What remains now is comparison of two fractions with numerator different and denominator different.

Note: No name has been given for such fraction.

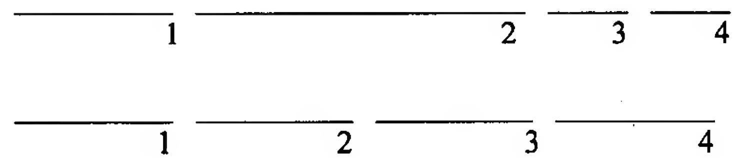
For teachers:

A similar thing is there in naming quadrilateral; when diagonals bisect each other at right angles, the quadrilateral

is a rhombus. When diagonals are such that they are perpendicular to each other; only one being bisected, the quadrilateral is a kite. When the diagonals cut each other at right angles, the quadrilateral has not been given a name.

It is necessary to know the distinction between counting and measuring.

A look at the diagram given below:



Brings out the distinction.

The top rows of segments are of different lengths and so the numbers give the count of segments.

The bottom row consists of segments of equal length. So the numbers not only count but also measures as a number of them taken together end to end has a length which is a certain number of times, one of them. Each one of them becomes a unit of measurement of length.

Note:

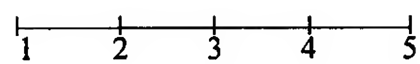


fig (i)

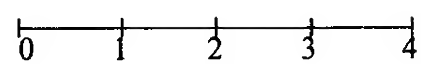


fig (ii)

The numbers in fig (i) are counting ones. Parts are counted. The numbers in fig (ii) are measuring ones, equal segments are measured. Note the start from 0.

§ Operation of addition.

Addition of like fractions:

Eg.

(i) $\frac{4}{15}+\frac{7}{15}=\boxed{}$

What does this mean? There are 4 fifteenths followed by 7 fifteenths. Together there are 11 fifteenths.

$$\text{So, } \frac{4}{15}+\frac{7}{15}=\frac{11}{15}$$

It is often presented as $\frac{4+7}{15}=\frac{11}{15}$

So, in the case of like fractions, numerators are added with the denominators unchanged, the sum being a like fraction.

(ii) $\frac{1}{8}+\frac{5}{8}=\boxed{}$

1 eighth + 5 eights = 6 eighths

$\frac{1}{8}+\frac{5}{8}=\frac{6}{8}$, that it has $\frac{1+5}{8}=\frac{6}{8}$

[Note: Teachers and textbook writers for that matter insist on simplifying the sum fraction if it permits of. This is not necessary, unless the question is given to be not simply add, but give the sum in the simpler form.]

Here $\frac{6}{8}=\frac{3}{4}$ so $\frac{1}{8}+\frac{5}{8}=\frac{3}{4}$

(iii) $\frac{3}{10}+\frac{9}{10}=$

$$\frac{3}{10}+\frac{9}{10}=\frac{12}{10}=\frac{6}{5}=1\frac{1}{5}$$

Again while giving the answer in situation like this, it is expected to be given as mixed fraction, which consists of whole part and fraction part. Again this need not be required unless it is specifically asked to be given as mixed fraction or the context warrants it.

**** Conversion of a mixed fraction into improper fraction:***

$$3\frac{4}{7}=$$

$$3\frac{4}{7}=3+\frac{4}{7}$$

$$=\frac{21}{7}+\frac{4}{7}=\frac{21+4}{7}=\frac{25}{7}$$

(It is usual to see teacher instructing children to do this by multiplication together with addition divided by the denominator given as shown below:

$$3\frac{4}{7}=\frac{3\times7+4}{7}=\frac{25}{7}$$

It is common to find children doing it correctly without knowing why it is done so).

Addition of two mixed fractions:

Find the sum:

$$5\frac{2}{3}+7\frac{3}{4}$$

There are two ways of doing this:

(i) Adding wholes to wholes, fractional part to fractional part and then adding them to give the sum:

$$5\frac{2}{3}+7\frac{3}{4}$$
$$=5+7+\left(\frac{2}{3}+\frac{3}{4}\right)$$
$$=5+7+\left(\frac{8+9}{12}\right)$$
$$=12+\frac{17}{12}$$
$$=12+1\frac{5}{12}$$
$$=13\frac{5}{12}$$

(ii) Converting each to improper fraction and adding the improper fractions:

$$5\frac{2}{3}+7\frac{3}{4}$$
$$=\frac{17}{3}+\frac{31}{4}$$
$$=\frac{17\times4+3\times31}{12}$$
$$=\frac{68+93}{12}=\frac{161}{12}=13\frac{5}{12}$$

Operation of Subtraction:

It is similar to operation of addition. Subtraction of like fraction is thus skipped.

Subtraction of a fraction for a whole number needs attention.

(i) $1-\frac{5}{8}=\frac{8}{8}-\frac{5}{8}=\frac{8-5}{8}=\frac{3}{8}$

(This can be done mentally without steps by taking the difference between the denominator and numerator as numerator with denominator remaining the same).

(ii) $3-\frac{7}{9}=2+\left(1-\frac{7}{9}\right)=2\frac{2}{9}$

(iii) $4-1\frac{3}{4}=(4-1)-\frac{3}{4}=3-\frac{3}{4}=\frac{12-3}{4}=\frac{9}{4}=2\frac{1}{4}$

or $3+1-1-\frac{3}{4}=3-\frac{3}{4}=2+1-\frac{3}{4}$

$=2\frac{1}{4}$

(iv) $3\frac{1}{4}-2\frac{5}{8}=(3-2)+\left(\frac{1}{4}-\frac{5}{8}\right)$

$=1+\left(\frac{2}{8}-\frac{5}{8}\right)$

$=\frac{8}{8}+\frac{2}{8}-\frac{5}{8}=\frac{5}{8}$

or $2+1\frac{1}{4}-2\frac{5}{8}$

$=(2-2)+\left(1\frac{1}{4}-\frac{5}{8}\right)$

$=\frac{5}{4}-\frac{5}{8}=\frac{10}{8}-\frac{5}{8}=\frac{5}{8}$

(v) $3\frac{2}{3}-1\frac{7}{8}$

$=(3-1)+\left(\frac{2}{3}-\frac{7}{8}\right)$

$=2+\left(\frac{16-21}{24}\right)$

$=2-\frac{5}{24}=1+\frac{24}{24}-\frac{5}{4}$

$=1\frac{19}{24}$

§ Operation of multiplication:

(i)

$\frac{2}{3}\times 4=2\frac{2}{3}$

or $\frac{8}{3}$

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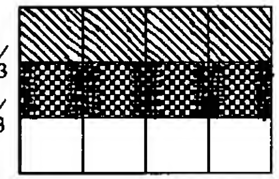

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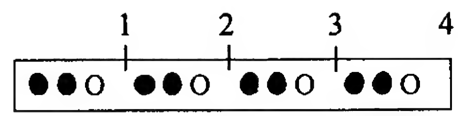
Continuous situation

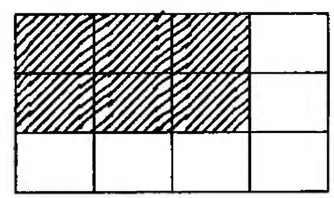


Discrete situation

$\frac{2}{3}\times 4=\frac{8}{3}$ or $2\frac{2}{3}$

(ii) $\frac{2}{3}\times\frac{3}{4}=\frac{6}{12}$

⅔



Continuous situation

⅓

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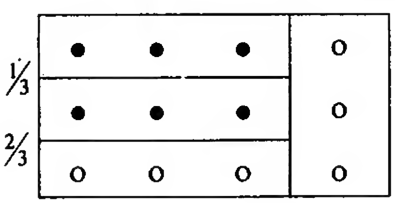
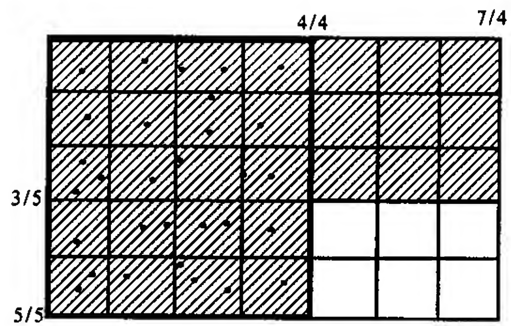


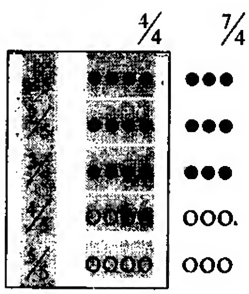
¾ 6⁄12

Discrete situation.

(iii) $\frac{3}{5} \times \frac{7}{4} = \frac{21}{20}$



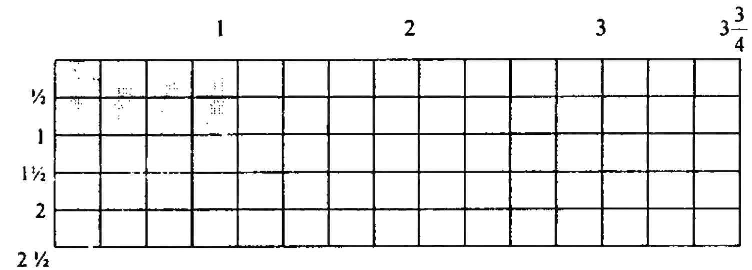
Continuous situation.

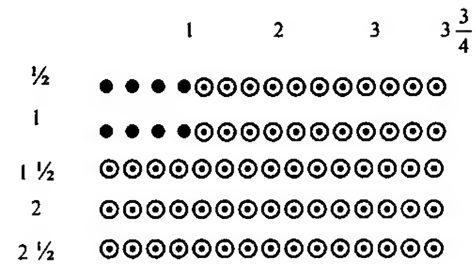


Discrete situation.

Note: setting up discrete situation requires ingenuity and it is not as easy as continuous situation. Paper folding needs to be done for continuous situation and bottle tops used for discrete situation (or some other suitable material).

(iv) $2\frac{1}{2} \times 3\frac{3}{4} = \frac{75}{8} = 9\frac{3}{8}$





$$2\frac{1}{2} \times 3\frac{3}{4} = \frac{75}{8} = 9\frac{3}{8}$$

Note: All these visualizations or concretizations are not to be done by children. It is enough if the teacher sets them up and ask children to discover the rule.

Rule:

Product of two fractions is

$$\frac{\text{Product of numerator}}{\text{Product of denominator}}$$

Once the rule is discovered, children can do multiplication of two fractions, both like, both unlike one of them whole number and the other a fraction. One of them is a mixed number and the other fraction.

One of them is mixed and the other is also mixed. Extending the product of more than two fractions is similar to extension of product of more than 2 whole numbers.

Operation of division:

Fractions have been introduced to make division a closed operation.

This can be presented dramatically as follows:

0	3/4	1	7/5
4) 3	4) 3	5) 7	5) 7
0	3	5	7
3	0	2	0

Continuous situation requires paper folding and discrete situation setting up objects. These practicals enable children to get at the rule of 'invert and multiply', in doing division with fraction. The pace has to be gradual as children will have to see their ways.

First of all, the meaning of division has to be understood, as *division* is a technical word,

Instead of ÷ sign, --- is used.

What does $\frac{12}{3}$ (12 division by 3) mean?

Since 3 is less than 12, it means finding number of 3's in 12.

What does $\frac{3}{12}$ mean? It cannot be number of 12's in 3, as 12 is greater than 3. it means finding what part of 12 is 3.

With this understanding, children are ready to launch on division venture and discover the rule.

There is a preparatory experience requiring the concept of reciprocal.

What is $\frac{3}{4} \times \frac{4}{3}$? it is 1.

Illustration:

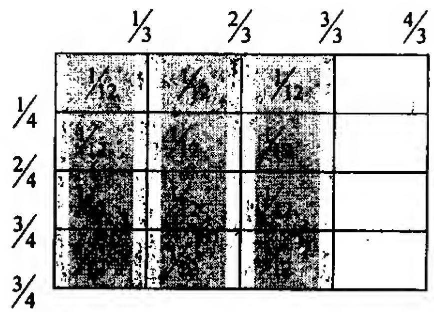
Discrete situation:

$$\frac{3}{4} \times \frac{4}{3} = \frac{12}{12} = 1$$

	1/3	2/3	3/3	4/3
1/4	● 1/12	● 1/12	● 1/12	o
2/4	● 1/12	● 1/12	● 1/12	o
3/4	● 1/12	● 1/12	● 1/12	o
3/4	● 1/12	● 1/12	● 1/12	o

Continuous situation:

$\frac{3}{4} \times \frac{4}{3} = 1$

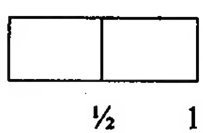


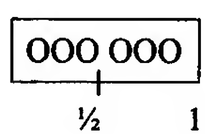
Children discover that a fraction multiplied by another fraction which is the inverted form of the fraction taken (numerator and denominator of one gets interchanged when inverted) gets back the whole, which is 1. Introduce the vocabulary reciprocal. A reciprocal of a fraction is its inverted form. Every fraction except zero has a reciprocal. Since every natural number is a fractional number with 1 in the denominator, it has a reciprocal.

Eg. Reciprocal of 3 is $\frac{1}{3}$ since 3 means $\frac{3}{1}$.

Reciprocal of 1 is itself since inverted form of $\frac{1}{1}$ is $\frac{1}{1}$.

(i) What is $1 \div \frac{1}{2}$? Continuous discrete



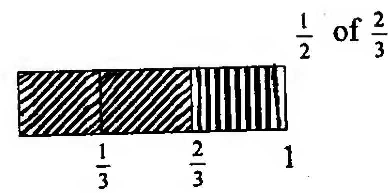


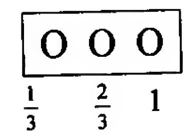
How many $\frac{1}{2}$'s are there in 1? There are 2 halves in 1.

Similarly $1 \div \frac{1}{3} = 3$

$1 \div \frac{1}{4} = 4$ and so on.

(ii) What is $1 \div \frac{2}{3}$





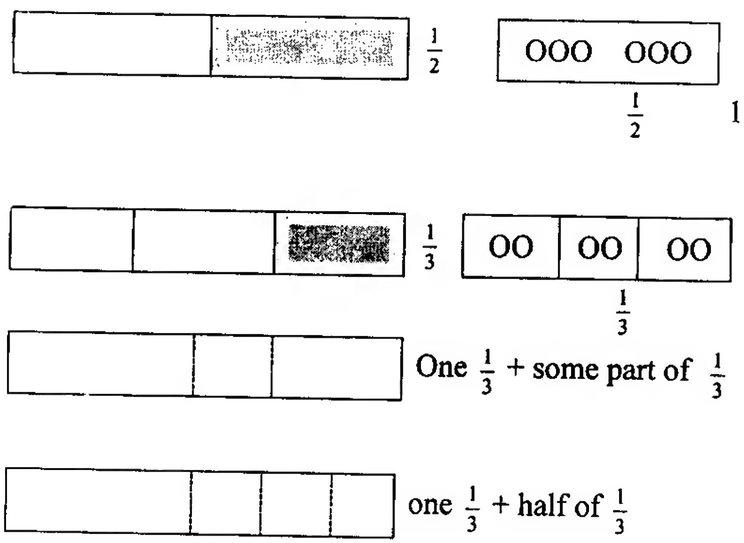
$1 \div \frac{2}{3}$ = one $\frac{2}{3}$ and half $\frac{2}{3}$ or $1\frac{1}{2}$ two thirds.

Similarly $1 \div \frac{3}{4} = \frac{4}{3}$ and $1 \div \frac{5}{8} = \frac{8}{5}$ and so on.

(iii) What is $\frac{1}{2} \div \frac{1}{3}$

Since $\frac{1}{3}$ is less that $\frac{1}{2}$, the question is to find how many $\frac{1}{3}$,s are there in $\frac{1}{2}$.

Continuous situation: ***Discrete situation:***

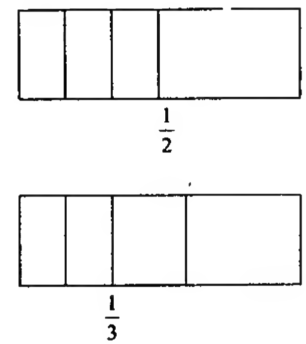


This concretization helps visualization of $\frac{1}{2} \div \frac{1}{3}$ as one $\frac{1}{3}$ and half of $\frac{1}{3}$.

The same set up is enough to find $\frac{1}{3} \div \frac{1}{2}$ meaning what part of $\frac{1}{2}$ is $\frac{1}{3}$. Since $\frac{1}{2}$ is greater than $\frac{1}{3}$.

The visualization gets the response

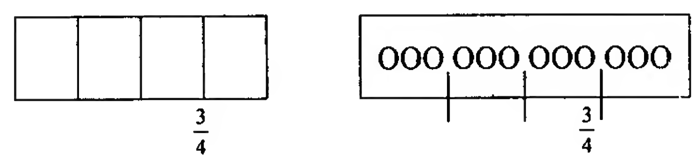
$$\frac{1}{3} \div \frac{1}{2} = \frac{2}{3}$$

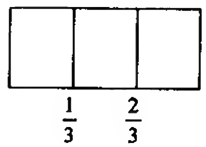


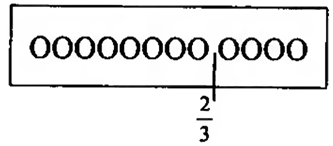
$\frac{1}{2}$ is taken as a whole and since each object is $\frac{1}{3}$.

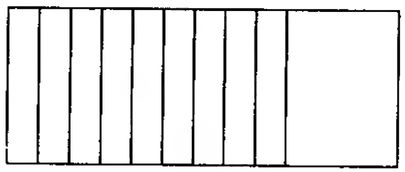
$$\frac{\frac{1}{3} \textit{ of the whole}}{\frac{1}{2} \textit{ of the whole}} = \frac{2}{3}\left(\textit{of } \frac{1}{2}\right)$$

(iv) what is $\frac{3}{4} \div \frac{2}{3}$?









$$\frac{3}{4}\div\frac{2}{3}=1\frac{1}{8} \quad or \quad \frac{9}{8}$$

$$\frac{3}{4}\div\frac{2}{3}=\frac{3}{4}\times\frac{3}{2}$$

also $\frac{2}{3}\div\frac{3}{4}=\frac{8}{9}=\frac{2}{3}\times\frac{4}{3}=\frac{8}{9}$, a fraction.

This shows that dividing a fraction by another is simply the fraction multiplying by the reciprocal of the dividing fraction.

Note: In earlier times, the rule 'invert and multiply' was given

(1) By the teacher and children did exercises after exercises in getting the rule remembered for use in division of fraction. Now-a-days it is stated that a division of a given fraction by another is nothing but the given fraction multiplied by the reciprocal of the dividing fraction 'teach by rule'.
In short 'invert and multiply' of earlier times is taken as 'multiply by the reciprocal' to suit to the modern times when concepts have to be got first.

(2) Use of continuous situation and use of discrete situation in these practicals for division of fractional number have each its inherent difficulties whereas in continuous situation

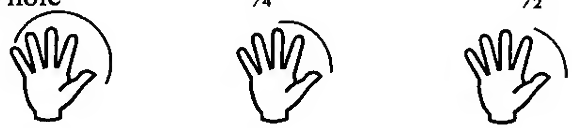
requiring rectangle paper folding, setting up is easy but to get at the outcome is tricky, in discrete situation consisting of separate objects, setting up is tricky but reading the outcome is easy.

(3) Children are not to have this in exam. When paper folding and dots display are made before hand for students to suggest the division of fractional numbers with reading of outcome.

(4) There should be a prominent place in annual school math exposition when children and parents alone, of the exposition holding school, with hands on – take off live demo presentation, would go round and learn how concepts are evolved in mathematics. Present day exhibition, which charts splash all over are just showing to build up the image of the school; neither the children belonging to the exhibition conducting school, nor the visiting children from outside have time and patience to understand. The students in charge are volunteers picked up at the last moment with a little briefing. Photos are taken and they find a place in the school annual published or preserved in an album if it is the practice and in news media sometimes.

(5) It is natural to expect the use of fingers in presenting divisions of fractions and getting the rule of invert and multiply or the property of multiplication by the reciprocal.
An illustration is given here to show the possibility

Whole $\frac{3}{4}$ $\frac{1}{2}$

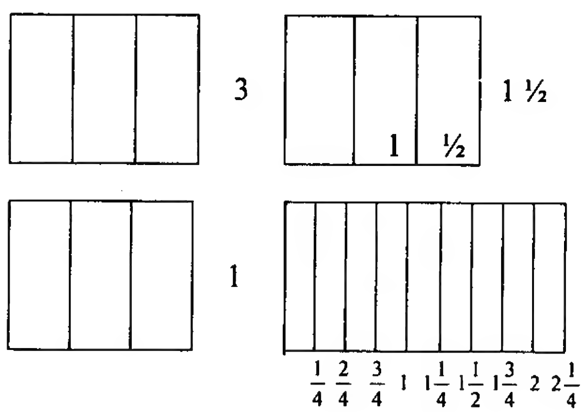


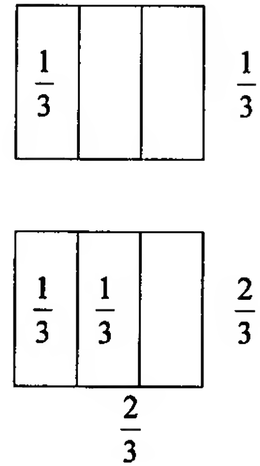
How many ½ 's are in ¾ .

To reinforce the understanding that whole and part are relative, give children three rectangular sheets of paper of uniform shape and size and pose the question whether by using them the folding can be exhibited. Those children who are able to respond to see their challenge rewarded are endowed with intuition to see their way.

$3, 1, \frac{1}{3}, 2\frac{1}{4}, 1\frac{1}{2}, \frac{2}{3}$ and so on.

Responses that have been obtained are shown below.





For indepth understanding of *whole* and *part*, a project such as given below can be found to fulfill the objective. Fill in the rest.

OOO	1	-	-	-	-
OOOOO	$\frac{5}{3}$ or $1\frac{2}{3}$	1	-	-	-
OO	$\frac{2}{3}$	-	1	-	-
OOOO	$\frac{4}{3}$	-	-	1	-
OOOOOO	$\frac{6}{3}$ or 2	-	-	-	1

When objects constitute a whole, how will the rest be compared? The first column is worked out. The rest can be worked out in the similar way.

Instead of discrete set, strips of paper indicating the fraction could also be used. But this requires greater care, the size of the fractional part in each strip be the same.

If a child were to come out with the affirmation that any fraction can be shown, that child is certain to be bright. It would be ideal that children use rectangular sheets of paper of different sizes, as that would help abstraction of whole and part.

Note: It will be difficult to show $\frac{1}{5}, \frac{1}{7}$, etc by paper folding. One way is to get a paper folding to show 6ths and then detach 1 part and treat the remaining as a whole to show fifths. So is the case to get $\frac{1}{7}$ by taking eighths and removing one eighths and treating the remaining portion to represent the whole to show sevenths.

Paper folding is not what is to be emphasized but visualisation of fraction and reaching take off stage. Consecutive unit fractions that can be shown by folding are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{12}, \frac{1}{16}$

(9) PRIME FACTORISATION, LCM AND HCF

Syllabi in all systems of schools cover this topic of prime factorization; LCM and HCF but the treatments are somewhat varied and disconnected.

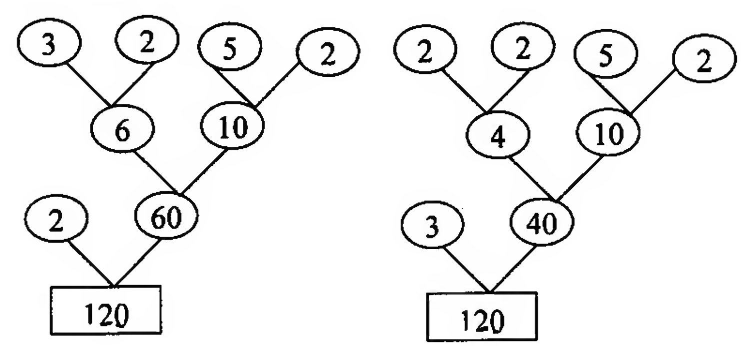
Consider 120. If asked to be expressed as a product of two factors the response is not unique.

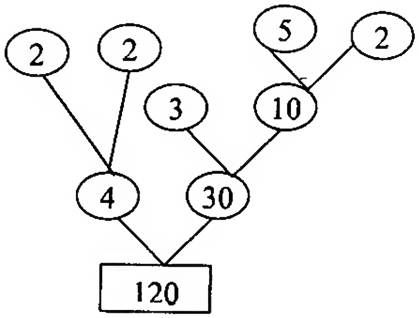
$$\begin{aligned} 120 &= 2 \times 60 \\ &= 3 \times 40 \\ &= 4 \times 30 \\ &= 5 \times 24 \\ &= 6 \times 20 \\ &= 8 \times 15 \\ &= 10 \times 12 \end{aligned}$$

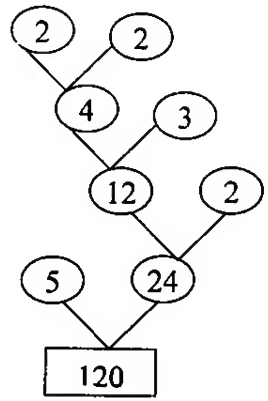
The factors can both be composite, one composite and the other prime, or both prime.

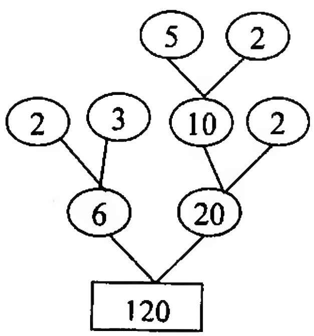
(Eq. $120 = 10 \times 12$ (both composite), $45 = 3 \times 15$ (one composite and the other prime), $21 = 3 \times 7$ (both prime)).

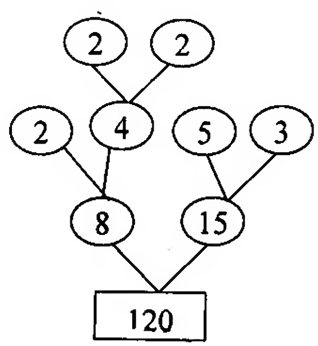
If it is required to have factor, all prime, the composite factor that appears need to be factorised further. This is arrestingly by means of factor trees as pictured hitherto:

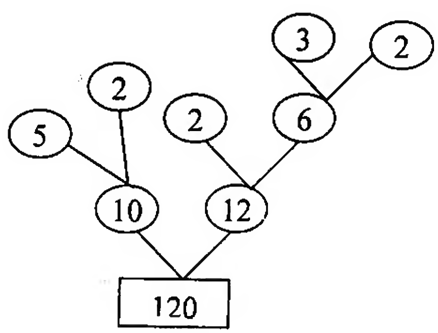












Once prime numbers are reached as factor, factorization is seen to end. Children could notice that the same prime factor appear at the stopping stage though their order of appearance is not the same. If order is taken into account, the ultimate factorization turns out to be unique – a magic property of numbers, central to number theory.

$$120 = 2 \times 2 \times 2 \times 3 \times 5$$
$$= 2^3 \times 3 \times 5$$

DISCOVERY:

Prime factorization of any number is unique (meaning 2 repeated thrice, 3 occuring once and 5 occuring once.)

LCM:

When two numbers are given, they have their least common multiple called in short as LCM.

This requires listing of multiples of two numbers separately, identifying the common multiples and fixing the least of the common multiples.

An example will explain:

Take 10 and 12.

Multiples of 10 are 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 120, 120, and so on.

Multiples of 12 are 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, and so on.

Common multiples of 10 and 12 are 60, 120, 180 and so on (noting but multiplication table of 60 times).

The least of the common multiples is 60.

So LCM of 10 and 12 is 60.

Prime factorization of 60:-

This is rather lengthy. If their prime factorizations are compared, LCM can be identified faster.

Prime factorisation of 10 and 12.

$$10 = 2 \times 5$$
$$12 = 2 \times 2 \times 3$$

Using them $2 \times 2 \times 3 \times 5$ has to be fixed to get at their LCM 60.

$$10 = 2 \times 5$$
$$12 = 2 \times 2 \times 3$$

The common factor 2 is taken out and the next of the factors, which are not common, are also taken, giving the prime factors of the LCM of the two numbers. So LCM here is

$$2 \times 2 \times 3 \times 5 = 60.$$

There is a division method that secures the L.C.M with less number of steps. But the division is not an ordinary one. So there is chance for confusion. This should be guarded against.

Preceding the introduction of division method. LCM of two numbers in situations needs to be taken up and familiarity obtained.

(1) 8 and 15
$$8 = 2 \times 2 \times 2$$
$$15 = 3 \times 5$$

L.C.M. of 8 and 15 is $2 \times 2 \times 2 \times 3 \times 5$ or 120

Since 8 and 15 are relatively prime, the LCM can be given by getting their product.

(2) 6 and 18
$$6 = 2 \times 3$$
$$18 = 2 \times 3 \times 3$$

$\therefore$ L.C.M of 6 and 18 is $2 \times 3 \times 3$ or 18.

Since 18 is a multiple of 6, 18 is the LCM of both.

(3) 10 and 25.
$$10 = 2 \times 5$$
$$25 = 5 \times 5$$

LCM of 10 and 25 is $5 \times 2 \times 5 = 50$.

Since 10 and 25 are such that are not coprime, neither the greater one is the multiple of the smaller one. In such cases, LCM has to be found.

Note: In case of small numbers less than10 to start with and then extended to numbers less than 20, LCM can be found by mental calculation as instanced below:

When two numbers are given, it can be seen that the common multiple, which is the LCM, occurs earlier in multiples of the greater number.

So to hit upon the LCM run though the multiples of the greater number in order, checking every time whether it is divided by or a multiple of the smallest. The first such one is the LCM.

Examples: (i) LCM of 6,8
Multiplies of the greater number 8 is

8	16	24
×	×	✓

so 24 is the LCM of 6 and 8.

(ii) LCM of 8,10

Multiples of the greater number 10 is

10,	20,	30,	40
×	×	×	✓

So 40 is the LCM of 8 and 10.

Division Method:

(i) LCM of 10 and 25.

$$\begin{array}{r|l} 5 & 10,\ 25 \\ \hline & \ \ 2,\ \ 5 \end{array}$$

(note 2&5 are coprime)
so LCM of 10 and 25 is
$5 \times 2 \times 5 = 50$

(ii) LCM of 24 and 40.

$$\begin{array}{r|l} 2 & 24,\ 40 \\ \hline 2 & 12,\ 20 \\ \hline 2 & \ \ 6,\ 10 \\ \hline & \ \ 3,\ \ 5 \end{array}$$

So LCM of 24&40 is $2 \times 2 \times 2 \times 3 \times 5$ $=120$.

It is safer to go in order of primes.

The division can be shortened if care is taken.

$$\begin{array}{r|l} 8 & 24,\ 40 \\ \hline & \ \ 3,\ \ 5 \end{array}$$

L.C.M is $8 \times 3 \times 5 = 120$.

LCM of three or more numbers:

More experience of finding LCM of two numbers is not enough to meet the problems arising in getting the LCM of three or more numbers.

Consider 10,12 and 15.

M_{10} or $M_{(10)}$: 10,20,30,40,50,60,70,80,.......
M_{12} or $M_{(12)}$: 12,24,36,48,60,72,........
M_{15} or $M_{(15)}$: 15,30,45,60,75,.....

In writing out the sequence of multiples of numbers given, there is no stipulation about the number of multiple sequences have to be extended wherever necessary till common multiples of (10,12 & 15) or ($M_{10,12,15}$)
Is 60.

Going by prime factorsation method,

$$\begin{aligned} 10 &= 2 \times 5 \\ 12 &= 2 \times 2 \times 3 \\ 15 &= 3 \times 5 \\ 60 &= 2 \times 2 \times 3 \times 5 \end{aligned}$$

LCM shows common factor 2 is taken though it is not common to all the three, but only to two of them, common factor 3 is taken like wise and finally common factor 5 taken like wise.

There is one 2 which is not had in the common factor. So 2 is also to be taken

Now

$$\begin{aligned} 10 &= 2 \times 5 \\ 12 &= 2 \times 2 \times 3 \\ 15 &= 3 \times 5 \end{aligned}$$
LCM of 10,12,15 $= 2 \times 3 \times 5 \times 2$
$= 60$.

Division method in getting LCM of three or more number has its peculiarities, which required to be noticed, and familiarded.

$$\begin{array}{r|l} 3 & 10,\ 12,\ 15 \\ \hline 2 & 10,\ 4,\ 5 \\ \hline 5 & 5,\ 2,\ 5 \\ \hline & 1,\ 2,\ 1 \end{array} \qquad \begin{array}{r|l} 2 & 10,\ 12,\ 15 \\ \hline 3 & 5,\ 6,\ 15 \\ \hline 5 & 5,\ 2,\ 5 \\ \hline & 1,\ 2,\ 1 \end{array} \qquad \begin{array}{r|l} 5 & 10,\ 12,\ 15 \\ \hline 2 & 2,\ 12,\ 3 \\ \hline 3 & 1,\ 6,\ 3 \\ \hline & 1,\ 2,\ 1 \end{array}$$

So LCM of 10, 12, 15 is $2 \times 2 \times 3 \times 5 = 60$.

While finding LCM of more than two numbers, if one of them is a factor of one of the other two or both the two other, then it can be ignored.

E.g:-

The LCM of 4,6 and 8. Since 4 is a factor of 8, 4 can be ignored, and the LCM of 6 and 8 can be taken into account.

HCF (Highest common factor).

The number of factors of a number are finite. Since every counting number is divisible by 1 and the number itself, they are not to be emphasized while listing factors.
HCF of 12 and 42.

Factors of 12 : 1,2,3,4,6,12
Factors of 42: 1,2,3,6,7,14,21,42

Common factors of 12 and 42 are 1,2,3.
The highest of this is 6. so 6 is the HCF of 12 and 42.

The HCF of 12 and 42 through prime factorisation.

$$12 = 2 \times 2 \times 3$$
$$42 = 2 \times 3 \times 7$$

common factors of 12 and 42 is 2 and 3 and the highest of them is 2×3 or 6.

(*Note* : HCF is also called GCM).

HCF through division method:

$$\begin{array}{r|l} 2 & \underline{12, 42} \\ 3 & \underline{6, 21} \\ & 2,\ \ 7 \end{array}$$

Since 2 and 7 are coprime, HCF is 2×3 or 6.

- in the case of small numbers such as digits and some numbers less than 20, the HCF of two such numbers can be determined mentally.

Since the HCF occurs first in the factors of the smallest number, one has to run through the sequence of factors of the smaller number, checking at every stage if it is factor of the greater number also and stop when that happens.

That is the HCF of two numbers.
Find the HCF of 16 and 20.

16 is smaller, its factors are 2,4,8,16. Checking at every factor if it is a factor of the other number also, stop when the higher one is reached. Here it is 4, so 4 is the HCF of 16 and 20.

- If the two numbers are that one is a factor of the other, that factor number becomes their HCF.

  E.g.

  9 and 45.

9 is a factor of 45. So 9 is the HCF of 9 and 45. Let the child check and see if it is so by listing them.

$F_{(9)}$ or F_9 : 3,9
$F_{(45)}$ or F_{45} : 3,5,9,15

If two numbers are such that they have no common factors, they as has already been, if it is to be so called, introduced are coprime. Their HCF is 1.

HCF of three or more numbers:

If more then two numbers are taken, their HCF can be formed by listing method. But it is slow.

E.g. Find the HCF of 16,40 and 56.
F(16) or F16 : 2,4,$\underline{8}$,16
F(40) or F40 : 2,4,5,$\underline{8}$,10,20,40
F(56) or F56 : 2,4,7,$\underline{8}$,14,28,56

The highest factor common to the numbers is 8.
$\therefore$ HCF of 16,40 and 56 is 8.

- HCF can be formed by prime factor method

$16 = 2 \times 2 \times 2 \times 2$
$40 = 2 \times 2 \times 2 \times 5$
$56 = 2 \times 2 \times 2 \times 7.$

HCF is $8 = 2 \times 2 \times 2$ and it is got by taking only the factors common to all the three.

[*Note* : If it is to find the LCM, factors that are not common are also to be taken into account]

- HCF by division method.

$$\begin{array}{r|l} 2 & 16,\ 40,\ 56 \\ \hline 2 & 8,\ 20,\ 28 \\ \hline 2 & 4,\ 10,\ 14 \\ \hline & 2,\ \ 5,\ \ 7 \end{array}$$

The HCF is $2 \times 2 \times 2$ or 8. Since 2,5,7 have no common factor.

- One more example for finding HCF by division method is given to have better experience.

Find the HCF of 60, 90 and 75.

$$\begin{array}{r|l} 5 & 60,\ 90,\ 75 \\ \hline 3 & 12,\ 18,\ 15 \\ \hline & 4,\ \ 6,\ \ 5 \end{array}$$

Here, the HCF is $5 \times 3 = 15$, since 4,6,5 have no common factors.

Only when the given numbers are all divisible by a number, that common factor alone is considered unlike in division method adopted in finding LCM of the three numbers 60, 90 and 75, and see the difference.

$$\begin{array}{r|l} 3 & 60,\ 90,\ 75 \\ \hline 2 & 20,\ 30,\ 25 \\ \hline 5 & 10,\ 15,\ 25 \\ \hline & 2,\ \ 3,\ \ 5 \end{array}$$

The LCM is $3 \times 2 \times 5 \times 2 \times 3 \times 5 = 900$.

An important property of LCM and HCF:

$\text{H.C.F} \times \text{L.C.M} = N_1 \times N_2$.

This property is easily discovered by children where they draw up table and compare the two numbers with the LCM & HCF.

Let children complete the table and find the relation among them :

N_1	N_2	LCM	HCF
4	6		
6	8		
10	15		
12	16		
15	24		
8	12		
6	12		
9	18		

Children discover that product of two numbers is equal to the product of their LCM and HCF.

Once this property is got, it can be used to find LCM of two numbers easily.

For example: To find the LCM and HCF of two numbers 20, 24.

HCF of 20, 24 is 4.

So LCM is $\frac{20 \times 24}{4} = 120$.

HCF by division method:

When two numbers are given, their HCF can be found by a chain of division.

To find the HCF of 20,24:

```
20) 24 (1
    20
    ----
     4) 20 (5
        20
        ----
```

So 4 is the HCF.

What is the reasoning? It is as follows:
20,24 Does 20 divide 24? No.
Consider 24 – 20 and 20 to 4 and 20.
HCF of 4 and 20 should naturally divide 20 and 24.
Now 4 divides 20.
So 4 is the HCF of 20 and 24.

(10) DECIMAL FRACTIONS AND METRIC MEASURES

Introduction:

The topic decimal fraction is presented as a circus show of moving decimal point, removing decimal point and inserting it appropriately after the answer is got and so on.

It should be borne in mind that our ancients skipped this development. Though Chinese hit upon decimal fractions as the history facts, they did not develop it.

It was left to Simon Stevin of Belgium to present it and develop it and write a separate book on it called Le Thende (one tenth) in 1492. It took almost two centuries to get this move accepted and practised. There was hesitation in adopting metric system of measures of length, capacity and mass. There is continuing resistance to metric system of time owing to cultural resistance.

(Mr. Brij Bhushan Vij, member, Institute of Engineers (India), Individual member, Indian standard Institution and Fellow, Indian Association of History & Philosophy of Science wrote in 1982 a book on TOWARDS A UNIFIED TECHNOLOGY propagating metrication of Time and went about for some time crusading for it. It fell on deaf years and remains till today still born. For comparison, one could cite issue of National Calender or Rashtriya Panchanga published annually but not popular, though it was the outcome of the labours of the Central Govt. sponsored committee headed by the distinguished scientist M.N. Saha).

This theme needs to be taught with great care and imagination.

Consider the place values:

$$\begin{array}{ccccc} 10000 & 1000 & 100 & 10 & 1 \\ & \frac{10000}{10} & \frac{1000}{10} & \frac{100}{10} & \frac{10}{10} \end{array}$$

As place values are considered from units onwards in fixing higher units in sequence, it is easily seen that by taking 10 times higher unit in a place, next higher unit in the next place is fixed and this goes on non-stop.

Stop at some place; say showing the higher unit 1000. How is it obtained from its successive higher units? 10000 divided by 10 i.e. 10000/10 gives 1000. How is 100 obtained from 1000? 1000/10. How is 10 obtained from 100? 100/10. How is 1 obtained from 10? 10/10.

The question arises, 'why should it be stopped'. Thus Stevin raised the question. There is nothing to stop.

$$\begin{array}{lccccccccl} \ldots\ldots\ldots\ldots & 10000 & 1000 & 100 & 10 & 1 & \frac{1}{10} & \frac{1}{100} & \frac{1}{1000} & \text{and so on.} \\ \ldots\ldots\ldots\ldots & \frac{10000}{10} & \frac{1000}{10} & \frac{100}{10} & \frac{10}{10} & & & & & \end{array}$$

The lowest place values that get recognised are 1/10, 1/10 divided by 10 or 1/100, 1/100 divided by 10 or 1/1000 and so on.

It is interesting to note that Stevin could not suggest a suitable notation for writing this extension of place value to

the right of units. Getting a new idea is often faced by inability to give a suitably elegant notation. It was left to Napier, founder of logarithm to suggest the use of decimal point. How to read the number formed had to be faced. For e.g. how is 8375.269 read?

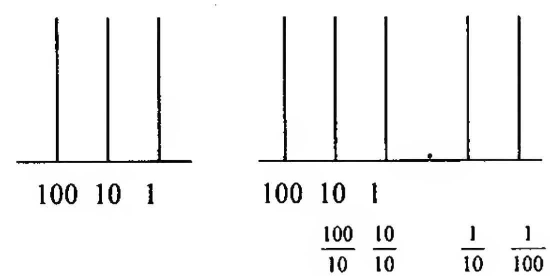
Either it is 8 thousand 3 hundred 75. Two six nine or 8375 and 269 thousandths.

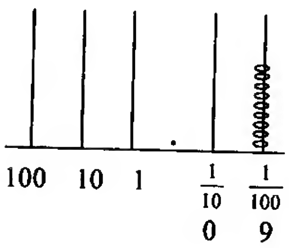
Note: Since children do not understand the significance of extension to introduce place value in fractional number, they are often seen to read the decimal part as simply 269 in the example chosen.

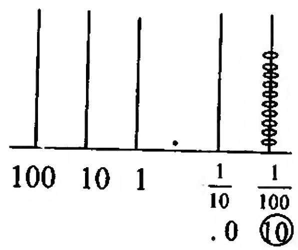
If 25/136 is considered, the fraction as such has no place value, though digits in numerator number and denominator number have place values.

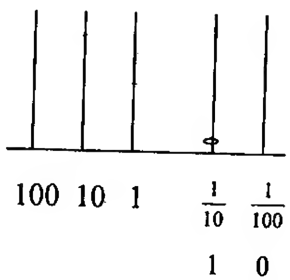
Use of abacus with removable spikes:

Visualization is helped if an abacus with removable spikes are used. Of course, no spike can have at most 9 rings. When one more is added they are to be removed and one ring inserted in the next higher place value.









The role of zero is spectacular when decimal fractions are handled.

Ordering decimal fraction

While ordering counting numbers, of two numerals, the one with greater number of digits is greater. This is not the case when decimal fractions are considered.

Compare .241 and .00356
.241 is greater than .000356.

Children have difficulty in understanding this and accepting it, because of mind set problem that gets formed while handling whole numbers.

If the concept of like and unlike decimal fraction is introduced, the comparison can be seen to disappear.

$$.241 \text{ is } \frac{241}{1000}$$

$$.00356 \text{ is } \frac{356}{100000}$$

These are unlike fractions. To make this like $\frac{241}{1000} = \frac{24100}{100000}$ has to be considered. And $\frac{241}{1000} = \frac{24100}{100000}$.

This enables one to see that $\frac{241}{1000}$ is greater than $\frac{356}{100000}$. that is to say .241 is greater than .00356.

This can be achieved by keeping the form of decimal fractions and annexing zeroes to have equal number of decimal places.

.241 and .00356.

When changed to like decimal fraction they read .24100 and .00356. now decimal point can be overlooked and comparison is made.

Children should be checked to see they distinguish between ordinary fraction and decimal fraction.

Let them be given the exercise of choosing the decimal fraction for the fractions given.

$$\frac{5}{8}, \frac{7}{10}, \frac{8}{15}, \frac{25}{100}, \frac{364}{1000}, \frac{10}{111}.$$

The fraction with denominator of higher units or power of 10 alone are decimal fractions and the rest are not. Incidentally, percentages are a decimal fraction having always 100 in the denominator.

Next comes the question of writing decimal fraction in the standard form. There should be emphasis on accounting for every place that removes cobwebs of confusion and misunderstanding dramatically.

Consider the conversion of decimal fraction given in to that in standard form.

(i) $\frac{7}{1000} = ?$

$$\frac{7}{1000} = \frac{0}{10} + \frac{0}{100} + \frac{7}{1000} = .007$$

(ii) $\frac{235}{10000}$ $\qquad$ $\frac{5}{10000}$ $\qquad$ (Start this way)

$$\frac{0}{10} + \frac{2}{100} + \frac{3}{1000} + \frac{5}{10000}$$

$=0.0235$

The usual practice is of instructing children to count the number of zeros and then count the digits in the numerator and fix the decimal point to the left and place the decimal point after accounting for number of places, putting zeros if there are no digits.

Operation:

A few typical cases are considered for each operation.

Addition:

237.8 + 41.762 + 5.3009

It is usual to instruct children to write the decimal fraction vertically with decimal point aligned.

$$\begin{array}{r@{.}l} 237 & 8 \\ 41 & 762 \\ 5 & 3009 \\ \hline 284 & 8629 \end{array}$$

Digits are added places column wise and the sum given with the decimal point fixed below the column of decimal fraction.

Note: When it comes to multiplication, children are instructed to write the fraction without decimal point and multiply the resulting numbers treating them as ordinary and finally fixing the decimal point after getting the total count of decimal places in both the numbers. It does work with considerable drill. But children wonder why decimal point is removed in multiplication but not in addition?

Teachers who remember their school days are at a loss to explain.

If like and unlike decimal fractions are introduced, addition as well as subtraction can also be done with decimal point removed.

Rewriting the addition sum with decimal fractions which are unlike not like ones, they become

2 3 7 8 0 0 0

4 1 7 6 2 0 Here given ten thousandths or $\frac{1}{10000}$'s

5 3 0 0 9

2 8 4 8 6 2 9

Writing this as a decimal fraction in the standard form the sum is 284.8629.

Subtraction:

Subtraction is also on similar lines. If irregular form is also used, drill gets considerably reduced and coverage is done with confidence by the teacher as well as the taught. Since this is rather unfamiliar to teacher and text book writer; teachers used to exert a little to get the required familiarity to handle this approach with confidence.

Some illustrations are given below:

Find the difference:

```
 5 0 3 8 . 0 0 3 1
-  4 7 9 . 6 2 4 8 2
```

Step one :

```
 5 0 3 8 . 0 0 3 1 0
-  4 7 9 . 6 2 6 8 2
```

changed to be like decimals.

Step two:

4	9	12	17	9	9	12	10	10
	4	7	9 .	6	2	6	8	2
4	5	5	8	3	7	6	2	8

changed to irregular from to familiarise subtraction without the botheration of borrowing

Note the case of doing subtraction from the left end instead of from the right end as is commonly the practice handed down by tradition.

Find the difference:

```
2 . 004
  . 855607
```

Step one:

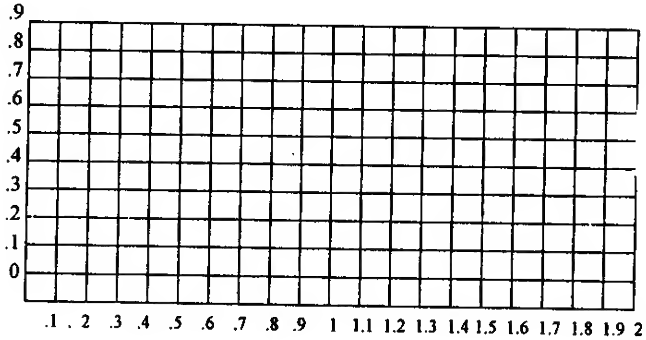
```
2 . 004000
0 . 855607
```

Step two:

1	9	9	13	9	9	10
0 .	8	5	5	6	0	7
1 .	1	4	8	3	9	3

Multiplication:

Use of 10 × 10 square ruled sheet can be seen to be welcomed by children to find the way on their own in doing multiplication of two decimal fractions (as well as division).



$1 \times .1 = .1$

Threat this ten by ten sequence of squares in the network showing whole and parts decimal by decimal.

Children can give the products given below in the background of their visuals:

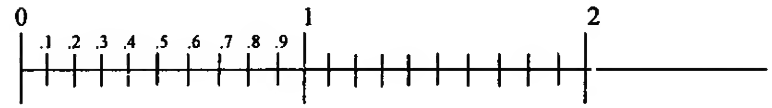
$$2 \times 0.2 = .4$$
$$.2 \times .2 = .04$$
$$.3 \times .2 = .06$$

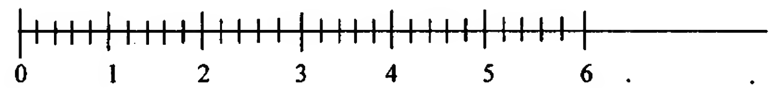
From the extended square network children can get products such as $1.2 \times 0.2 = 0.24$

Instead of the decimal fractions, the whole number situation should also be thought when $2\times 2 = 4$, $20 \times 2 = 40$, $12 \times 2 = 24$ can be identified.

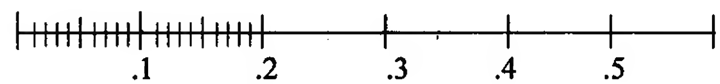
A foot rule showing inches each inch divided into 10 equal parts or cm and millmetres, representation of one decimal fraction can be visualized.

Concretization of decimal fractions with a graduated foot rule and a graduated metre slide:-



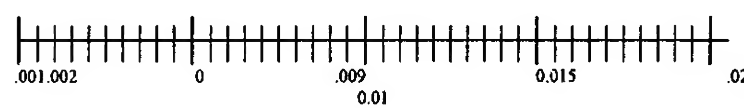


Further visualistion of hundredths and thousandths can be had by



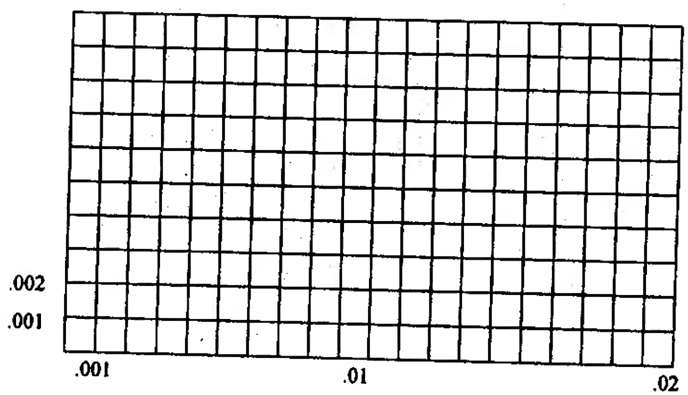
Treat 10cm as one. How do the readings change?

Take the metre scale. Take the meter as 1. What do the marking now read?



Thus conventional approach stirs imagination to respond with insight the development of decimal fraction.

In the same way, the 10 by 10 square network can also be used.



Though, concretization is not as direct as is had with considerations of markings as whole units, tenths and hundredths the inevitableness helps identifying hundredths and thousandths and ten thousandths. When asked in succession children see their way in getting the products.

$10 \times 2 = 20$
$10 \times 10 = 100$
$100 \times 1000 = 10000$

$.1 \times 2 = .2$
$.1 \times .1 = .01$
$.01 \times 10 = .1$
$.01 \times .01 = .0001$
$.01 \times .01 = .0001$
$.02 \times .02 = .0004$

$.001 \times .01 = .00001$
$.002 \times .02 = .00004$

$.01 \times 2 = .02$
$.01 \times .1 = .001$
$.001 \times 10 = .01$
$.001 \times .1 = .0001$

Setting the situation for division, the scale changes give the following:

$$\frac{20}{10} = 2 \qquad \frac{.2}{.1} = 2 \qquad \frac{.02}{.01} = 2$$
$$\frac{200}{10} = 20 \qquad \frac{2}{.1} = 20 \qquad \frac{.2}{.01} = 20$$
$$\frac{2000}{10} = 200 \qquad \frac{20}{.1} = 200 \qquad \frac{2}{.01} = 200$$
$$\frac{2}{.001} = 2000$$
$$\frac{.2}{.001} = 200$$
$$\frac{4}{2} = 2 \qquad \frac{.4}{.2} = 2 \qquad \frac{.04}{.02} = 2$$
$$\frac{.4}{2} = .2 \qquad \frac{.04}{.2} = .2 \qquad \frac{.004}{.02} = .2$$

Let children proceed on their own number line and so more divisions till they see the pattern of behaviour.

They are able to see that division with decimal fractions can be done by taking equivalent form:

$$\frac{1.2}{.6} = \frac{12}{6} = 2$$
$$\frac{1.2}{.03} = \frac{1.20}{.03} = \frac{120}{3} = 40$$
$$\frac{.12}{.6} = \frac{.12}{.60} = \frac{12}{60} = \frac{1}{5} = \frac{2}{10} = .2$$

also $\frac{.12}{.6} = \frac{1.2}{6} = .2$

It is seen that it is comfortable to have the division (or denominator) as a whole number in doing division with decimal fraction.

The rules about shifting decimal point to required number of places in multiplication or division get arrived at by children themselves.

§ Metric measures.

To facilitate communication, children need to be introduced to higher units 10, 100, 1000, etc and lower units $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$ linked with names in measures of metric system as shown below with abbreviation.

1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
kilo	hecta	deca	unit	deci	centi	milli
k	h	da	–	d	c	m

Once children get accustomed to he corresponding names of higher units in number system and metric measurers, they can frame their own tables of metric measures of length, capacity and mass (weight).

Situation	*name of unit*
Length	metre
Capacity	litre
Mass	gram (also gramme).

Taking length, the metric table of length is given as follows with abbreviations in accepted International usage.

10 millimetres (mm)	=	1centimetre (cm)
10 centimetres (cm)	=	1 decimetre (dc)
10 decimatres (dc)	=	1 metre (m)
10 metre (m)	=	1 decametres (dam)
10 decametres (dam)	=	1 hectametre (hc)
10 hectametres (hm)	=	1 kilometre (km)

Taking litre, the metric table of capacity, the table can be given by children themselves. Similarly by taking gram, the metric table of mass (weight) can be given by children themselves.

It is necessary that one more fact that 1000 kg makes / metric tonne needs to be familiarized.

Note: There are higher units and lower units of metric measures met with in scientific information. They are given below for teacher's reference.

Higher units:

10^6	-	1000000 Mega (M)
10^9	-	1000 000 000 Giga (G)
10^{12}	-	1000 000 000 000 Ten (T)
10^{15}	-	1000 000 000 000 000 Peta (P)
10^{18}	-	1000 000 000 000 000 000 Exa (E)

Lower units:

10^{-6} $\frac{1}{1000000}$ micro (μ)

10^{-9} $\frac{1}{1000000000}$ nano (n)

10^{-12} $\frac{1}{1000000000000}$ pico (p)

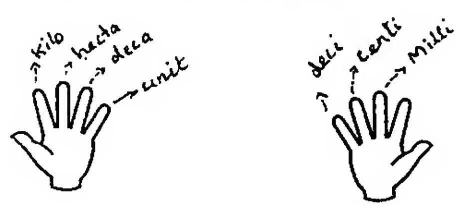
10^{-15} $\frac{1}{1000000000000000}$ femto (f)

10^{-18} $\frac{1}{1000000000000000000}$ atto (a)

When it comes to practical usage, a few higher units and lower units alone get mentioned and this should be made known to students.

Length km and m
Capacity l and ml
Mass (weight) kg and g.

To facilitate remembrance, these metric systems can be associated with fingers on both the hands.



Question of conversion of higher units to lower units and vice versa can be answered easily by having in mind their association of metric measures.

1 kilo = 1000 ***units*** = 1000000 ***milli***
1 unit = 1000 ***milli***
1000000 ***milli*** = 1 ***kilo***

A lot of time can be saved in covering the portion relating to metric measures by this preparatory warming up.

To help children to have ready measures is almost equal to the real ones, certain experiences can be brought to their notice.

Normally

Adult male palm (father's palm) when fingers are held together measures 10cm,

(But adult female palm (mother's palm) when fingers are held together measures 8 ½ cm)

The width of little finger is 1 *cm.*

The weight of a cubical container 1*cm* × 1*cm* × 1*cm* is 1 *litre.*

10*cm* × 10*cm* × 10*cm* = 1000 *litres*

Weight of a gem clip is 1 *gm.*

Note:

1. Children can wonder at nature's ways when they discover that one's height is equal to length between tips of middle finger of one's body with both the arms stretched..
   The table showing relation (almost approximate) between heights and weight may be had in the school, P.E. room, math lab/math activities centre, or even in class room. Weighing scales should also be provided. If we should copy westerners, it will help if we have bathroom scales.
2. Children should be encouraged to read the weight of the contents given in packaged food stuff, (powders, snacks, etc.) and the capacity of liquid contents in bottled commodities.

(11) DIVISIBILITY TESTS.

To avoid testing divisibility of a number by long division easier methods by inspection can be found by check divisor.

Invariably, divisibility tests are just stated without going into the rationale, thereby depriving children the golden opportunity to take a few steps in elements of introductory number theory.

There are a few prerequisites that children should know to launch on this formulation of tests of divisibility. If a number is divisible by another, the number is divisible by factor of the dividing number as well.

Partially, expanded notation in terms of place values comes handy.

Illustration:

$$
\begin{aligned}
5628 &= 562 \text{ tens} + 8\\
&= 56 \text{ hundreds} + 28\\
&= 5 \text{ thousand} + 628
\end{aligned}
$$

- When two (or more) numbers are multiples of a given number, then their sum is also a multiple of the given number.

Some Illustrations:

10 multiple of 2
(10 + 14) or 24 is a multiple of 2.
14 multiple of 2

In number theory symbols, 2/10 and 2/14 imply 2/(10+14)
21 is a multiple of 7
(So 21 + 56) or 77.
56 is a multiple of 7.

In number theory symbols, 7/21, 7/56, implies 7/(21+56). Stated differently, if two (or more) numbers have a common factor, their sum is divisible by the common factor.

* If a number is divisible by a factor, then a multiple of the number is also divisible by the factor.

Illustration

24 is a multiple of 8.
Then 13 × 24 is also a multiple of 8.
19 × 24 is also a multiple of 8.

In number theory symbols

8/24 implies 8/13 × 24 or 8/19× 24

- If the sum of two numbers is divisible by a factor, it does not follow that each of them is divisible by the factor.

Illustration:

(13 + 12) is a multiple of 5.

But 13 and 12 are not separately multiples of 5.

In number theory symbols:

"5/(13+12) does not imply 5/13 and 5/12" is true.

- If the sum of two numbers is divisible by a factor and if one of the two numbers is divisible by the factor, then the other is also divisible by the factor.

24 +56 is divisible by 8 and one of them 24 is divisible by 8. Then the other number 56 is divisible by 8.

In number theory symbols
8/(24 + 56) and 8/24 imply 8/56.
These can be elicited from children through discoveries from patterns.

- Before formulation tests, higher units or power of ten are to be studied with respect of divisibility of digits 1 to 9.

Divisibility by 2,4,8.

10	2 × 5
100	4 × 25
1000	8 × 125
10000	
100000	

The first higher unit divisible by 2 is 10. Since other higher units are multiples of 10, they are naturally divisible by 2.

The first higher unit divisible by 4 is 100; since other higher units are multiples of 100, they are naturally divisibly by 4.

The first higher unit divisible by 8 is 1000. Since other higher units are multiples of 1000, they are naturally divisible by 8.

Illustration:

Take the number 70486 (in base ten numeration)

- 70486 = 7048 tens + 6
  ten is divisible by 2 and more so 7048 tens. Since 6 is divisible by 2, the given number is divisible by 2.
- 70486 = 704 hundreds + 86.
  Hundred is divisible by 4 and more so 704 hundreds. But 86 is not divisible by 4. So the given number is not divisible by 4.
- Can we change the two digits to secure divisibility. Instead of 86, consider 84, for instance, which is divisible by 4. So 70484 is divisible by 4.
  (Instead of 86, there could be 84, 36, 16, 56, etc.)
- So what is the test of divisible by 4?
  If the number formed by the last two digits in a given number is divisible by 4, then the given number itself is divisible by 4.

  Often it can be seen in textbooks even today incorrect formulae of divisibility tests for 4 and 8.
- Instead of stating the number formed by the last two digits in the case of divisibility by 4 and number formed by the last three digits in the case of divisibility by 8, the phrase 'number formed is seen omitted.'

Consider 7228; of the last two digits 2 and 8, 2 is not divisible by 4, but 8 is but the number is divisible by 4 if taken as 28 which is 4×7.

Considering 936. Neither of the last two digits are divisible by 4, but the number is divisible by 4, as $36 = 4 \times 9$.

Similar situation prevails in the case of divisibility by 8. Considering 56216, none of the last three digits are divisible by 8 but the number is divisible by 8 as $216 = 8 \times 27$.

70486 = 70 thousands + 486.

Thousands is divisible by 8 and more so 70 thousands. Is 486 divisible by 8? 486 is not divisible by 8. So the given number is not is not divisible by 8.

* Can we change a few of the last tree digits if not all to secure divisibility by 8?

Instead of 486, take 488. This is divisible by 8. So 70488 is divisible by 8.

A few other possibilities are
480, 408, 464, 496, 472, etc.

Divisibility for 5, 10

The first higher unit that is divisible by 5 and 10 is 10, and so what is required to be examined is the digit in the unit place the digit can be either 5 or 0.

When it is zero, the number with that units place digit is divisible by 5 as well as 10.

When it is five, the number with that unit place digit is divisible by 5 only.

Divisibility by 3 and 9:

10
100
1000
10000

No higher unit is divisible by 3 or 9.

If the predecessor of the higher units are taken, they are divisible by 3 as well as 9.

So 10	as to be taken as	$9 + 1$
100		$99 + 1$
1000		$999 + 1$ and so on.

Illustration:

Consider the number given in base ten numeration.
25614

Its expanded notation is

$$2 \times 10000 + 5 \times 1000 + 6 \times 100 + 1 \times 10 + 4.$$

Writing each higher unit in relation to its predecessor, the notation will read.

$$2\times(9999 + 1)+5\times(999 + 1)+6\times(99 + 1)+1\times(9 + 1)+ 4.$$

[*Note:* study	OOOOOOO	OOOO OOO
the example	OOOOOOO	OOOO OOO
with visual	2×7	$2 \times (4 + 3)$

When it is the same as $2 \times 4 + 2 \times 3$ – this expansion is the *distributive property of multiplication over addition.*]

Which is

$$2 \times 999 + 2 + 5 \times 999 + 5 + 6 \times 99 + 1 \times 9 + 1 + 4.$$

Separating this to show the number divisible by 3 (or 9)
25614

$= (2 \times 9999 + 5 \times 999 + 6 \times 99 + 1 \times 9)$
$+ (2 + 5 + 6 + 1 + 4)$

The sum of the digits is 18 and it is divisible by 3(or 9). So the divisibility tests for 3 (or 9) gets stated thus. If the sum of the digits [in the numeral of a number in base ten numeration] is divisible by 3(or 9), the number itself is divisible by 3(or 9).

Note: In finding the sum of the digits, the sum itself can be subject to the same divisibility test. In the instance taken digit sum of 18 is
1 + 8 or 9 and it is divisible by 3(or 9)

While adding the digits, if 3 or 9 or pairs with sum 3 or 9 occur, they can be ignored. In the instance taken
2 + 5 + 6 + 1 + 4 with effective sum 9.

Note: A number divisible by 9 is divisible by 3 and not vice versa.

E.g. 15 is divisible by 3 but not 9.

- An interesting question arises about the reduced digit sum not being divisible by 3 (or 9). What number property does it give? It is seen to be the remainder that the considered number gives on division by 3 or 9.

Illustration:

- 101, here the digits sum is 2. so 101 is not divisible by 3 or 9.

3|301 133-2   9|101 11-2

- 132, here the digit sum is 6. so 132 is divisible by 3, but not by 9.

3|132 44-0   9|132 14-6

Divisibility by 6

6 is 2 × 3. If a number is to be divisible by 6, it should be divisible by 2 well as 3.

In other words, an even number (ending in 0,2,4,6 or 8) satisfying the digit sum test for divisibility by 3 is divisible by 6.

Divisibility by 11

It is usual to include divisibility test for 11 also in the syllabus and the test is rather arrestingly complex. To arrive at the test is an exciting experience which children should not be denied.

First of all, children should be helped to become familiar with multiples of 11.

99, 9999, 999999 are all multiples of 11.

11, 1001, 100001, 10000001 are all multiples of 11.

A multiple of 11 is either the predecessor or successor of the higher units. This can be displayed as shown below:

	10	S
P	100	
	1000	S
P	10000	

and so on.

This is a beautiful number.

Observation of pattern

When the higher units has odd numbers of zero, then the successor of that unit is divisible by 11.

In the other case, when the higher units has even number of zero, the predecessor of that higher unit is divisible by 11.

This sets the stage to have the expanded notation reflecting this alternate behaviour of 11 while dividing into higher units.

For Instance take 68,475.

68475
$= 6 \times 10000 + 8 \times 1000 + 4 \times 100 + 7 \times 10 + 5$
$= 6\,(9999 + 1) + 8\,(1001 - 1) + 4\,(99 + 1) + 7\,(11 - 1) + 5$
$= 6 \times 9999 + 6 + 8 \times 1001 - 8 + 4 \times 99 + 4 + 7 \times 11 - 7 + 5$
$= (6 \times 9999 + 8 \times 1001 + 4 \times 99 + 7 \times 11) + (6 - 8 + 4 - 7 + 5)$

On putting together the multiples of 11 and putting together the rest separately.

The second part is 0 and it is divisible by 11. Hence the number itself is divisible by 11.

Note: The second part may also give 11 or multiples of 11 in deciding divisibility of 11.

How to fix the digits to be added and the digits to be subtracted?

There are a few different ways of giving the method.

(i) + - + - +
6 8 4 7 5

(ii) 6 8 4 7 5
o e o e o

The digits in odd places are to be added and the digits in even places are to be subtracted and the difference should be tested for divisibility of 11.

(iii) The Difference between the sum of digits in odd places and the sum of digits in even places to be tested for divisibility by 11.

So the test for divisibility by 11 gets stated then: find the two sums got by taking alternating digits of the number given and if the difference of the sum is 0 or a multiple of 11, the given number is divisible by 11.

Divisibility by 12.

Since 12 is 6×2 or 4×3, how should the tests of divisibility be stated? In the case of divisibility test for 6, divisibility 3 and divisibility by 6 is ensured.

If a number is divisible by 6 and by 2, is the number divisible by 12? Some children seem to think that on the model of divisibility test for 6, this will work. They are not alive to the fact that divisibility by 6 means the divisibility by 2 has been listed. So the test will not work. But taking 12 as 4×3, if a number satisfies divisibility tests for 4 and then by 3, then the number is divisible by 12.

How are 4 and 3 related? They are relative primes. So this should be checked before applying combined tests for composite number.

Divisibility by 7

One signitant omission in giving tests of divisibility is divisibility by 7. The test is not simple and straightforward like the tests for digits 2,3,4,5,6,8,9,10 and 11. However, there is a joint test, in the sense, divisibility by 1001 ensures divisibility by 7, 11 and 13 as well since $7 \times 11 \times 13 = 1001$. As in divisibility by 11, mark off the digits of a given numeral in groups of 3 digits starting from right end.

Consider for instance the question about the divisibility of 23456789 by 7. Marking off in periods of 3 digits from the right, what is got is 234, 567, 89.

As in test for divisibility by 11, take 3 digit groups alternately add the 3 digit number and supply.
Note:

* $1001 = 7 \times 11 \times 13$
* $1001 - 1 = 1000$
1000,000,001-1 = 1000,000,000 and so on.

In short the rationale below grouping the digits in three can be stated a below:

10^{15}	10^{12}	10^{9}	10^{6}	10^{3}	ones

Examine the divisibility of 257 914 657 by 1001.

Writing it as a sequence of 3 digit groups, starting from the right end,
257, 914, 657 is got

Now as in divisibility by 11, addition and subtraction is done alternately from right end.

$257 - 914 + 657 = 0$ and 0 is divisible by 7 and so $257 - 914 + 657$ is also divisible by 11 as well as 13.

Exercises can be varied, some with simple answer and some open ended.

Some of the typical questions are given as illustration.

(i) Which of the following is divisible by 8
7564328,
95006

(ii) Which of the following is not divisible by 4.
25821
323008

(iii) Is 19714 divisible by 6?
Use the digits to form a number divisible by 3 alone.

(iv) Fill up the place to get a number divisible by 9.
(a) 7 – 8 6 4
(b) 5   2 – 2

Use the same digits to fix number divisible by 9.

(v) Fill up the places to get a number divisible by 9.
(a) 6 – 31 – 1
(b) 78 – – 4

Give the count of number that are possible in each case.

Divisibility by 9.

If the digital root of a numeral is 9, the number given by the numeral is divisible by 9. of course, in getting the digital root, the digit 9 and digits making up 9 can be ignored to get the digital root faster. Given a numeral with some specified digits, other numerals with the same specified digits can be got by changing their places for being divisible by 9.

Of these numerals there are some interesting behavioural patterns.

- Consider, say a four digit number, 4781. Is this divisible by 9. Since its digital root is 2, the number is not divisible by 9. But a number can be generated by difference between the original numeral and the reversed numeral.

$$\begin{array}{r} 4781 \\ 1874 \\ \hline 2907 \end{array}$$

Now the digital root of the difference is 0 and so the difference represents a number divisible by 9.

- Instead of getting the difference between the numbers given by a multi-digit numeral and its reverse, the difference between the numbers given by a multi-digit numeral and a numeral formed by interchange of one or more pairs of digits can also be seen to be divisible by 9.

For example, consider.

57283 (its digital root is 7 and so the number is not divisible by 9.)

Now consider the number got by interchanging say 5 and 8, 7 and 3 viz. 83257. Now find the difference between the two numbers got.

$$\begin{array}{r} 83257 \\ 57283 \\ \hline 25974 \\ \hline \end{array}$$

It is easily seen that number is divisible by 9 since the digital root is 9.

What a fascinating behaviour of numbers!

Note: If children of class V are exposed to pattern language, they can be exposed to establish through pattern language special cases of this digital property of numbers generating multiples of 9.

- $1000\,a + 100\,b + 10\,c + d$ (4 digits numeral in general)
  $1000\,d + 100\,c + 10\,b + a$ (4 digits numeral reversed.)

Difference $999\,a + 90\,b - 90\,c - 999\,d$
$= 999\,(a \sim d) + 90\,(b \sim c)$ – a multiple of 9.

- $10000\,a + 1000\,b + 100\,c + 10\,d + e$ (5 digit numeral in general)
  $10000\,e + 1000\,d + 100\,c + 10\,b + a$ (5 digits numberal with digits in 'ths' and ones places, 'ths' and t's places interchanged)

$9999\,a + 999\,b - 990\,d - 9999\,e$
$= 9999\,(a \sim e) + 990\,(b \sim d)$ a multiple of 9.

The most general cases can be taken up in middle school stage.

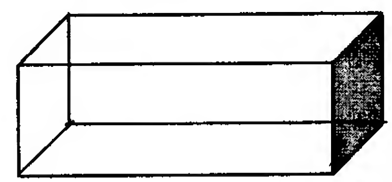
(12) SHAPES, CLASSIFICATION AND ANGLES

There are two approaches: start with recognition of solid shapes to be followed by recognition of plane shapes.

Start with recognition of plane shapes on surfaces of objects seen around reinforced by picture of them.

Empty rectangular boxes and cubic boxes can be got from medical shops, rectangular wooden pieces can be got from a carpenter's waste.

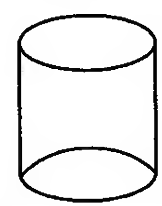
A rectangular box with lid / wooden piece



It has 12 edges, 6 faces and 8 corners.

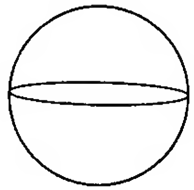
Each face is rectangular in shape. Edges are lines, line segment to be precise. Each corner is a point.

Cylindrical container with lid / wooden piece.



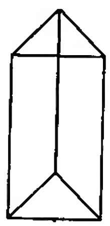
This has no corners. There are two circular ends around plane faces. The surface is curved.

Spherical ball



This has no corners, no faces. Curved surface is there all around.

It is within the capacity of children to fold a rectangular sheet of paper to form a cylinder, a circular sheet to fold into halves and quarters to identity the diameter and radius and discover that 2 times radius is 1 diameter.



Children can make triangular prism by folding a rectangular sheet of paper.

It has 9 edges, 2 triangular faces, 3 rectangular surfaces and 6 corners.

[By way of rousing the curiosity of children, children can make using plasticin, polyhedron with irregular faces, tabulate the number of edges, number of faces and number of corners of each polyhedron shaped as one's fancy dictates and discover a property which is independent of shape and size. Otherwise called topological property].

Polyhedron	No. of edges	No. of corners	No. of faces
I	12	8	6
II	9	6	5
III	15	10	7
IV	17	11	8

Ask the children to discover the number behaviour relating the number of corners and number of faces. It is a pleasant surprise that number of corners + number of faces – 2 = number of edges.

This topological property was stated by the swiss mathematician, Euler (1707 - 1753), who lived for mathematics, with a remarkable fund of discoveries to his credit.

This can be had in math club and math exposition.

* Earlier appreciation of point, line, line segment, plane, ray, angle and then representation, sets children on the thinking pad for handling mathematical concepts with confidence and care. This is one of the triumphs of productive mathematical thinking not covered in earlier decades of curriculum.

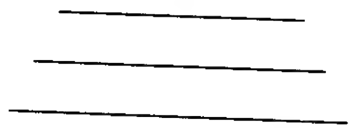
Through interactive involvement allowing permissible inaccuracies initially children get seized of the concepts.

Interactive session with students:

The teacher asks children to draw a line on the note book and asks them to draw a longer line. The teacher himself / herself draws a line on the black board. One of the children is invited to do it on the board.

________________ line
________________________ longer line.

The teacher exhorts them to draw a still longer line and asks another child to do it on the board.



Now the teacher gives them a challenging task for them to think and win.

'I think of a line such a way you cannot think of a longer line'. Can such a line exist? Can such a line be drawn?

One or two of the children care to respond that *the line should not have end points;* in case it has, a longer line can be thought of.

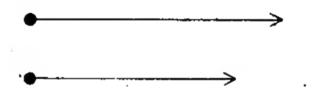
At this stage, children are ready to distinguish between a line and line segment. If points are there line segment is obtained if not it is a line and not a line segment. Whatever one draws is a line segment and not a line. How to draw a line is rather puzzling. Oly a cartoon can be drawn with arrows at either and to focus the attention of the viewer to the fact that it has no end points.

Line Line segment

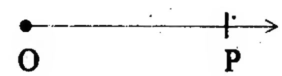
How to name this is the real question that gets raised.

A⟵——⟶B is *incorrect* as it looks it has and points named A & B. on the other hand, two points are to be chosen as A⟵|——|⟶B and named, the line getting name AB. Of course, line segment has its end points named P——Q.

Note: when a line segment is given there should be visualisation of the line of which it is apart, though it has to be shown by extending the line segment either way. It has length with measure. Line has length, which cannot be measured. What is a ray? Recall, rays from the SUN and rays from light. They have a beginning point and no end point.



These are rays. Their lengths cannot be measured. Since one end point is known, it is named; another point is selection on the ray, to get the ray named as shown hitherto.



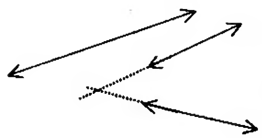
Ray *OP*.

A line can be visualized to have two rays in opposite ⟵——⟶ direction;

It is to be realized that since lines have measures of length their visuals may appear to differ but they represent the same line.

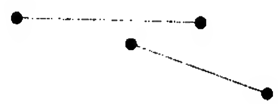
Note: Mathematicians have introduced three concepts to help visualization of sequence of number system; a line for integers, rational numbers and real number and a ray for whole numbers positive fractions, positive rational numbers and positive real numbers. At the same time, the distinction between definition and description is made clear.

Now consider a plane extending on all sides, that is to say having no boundary. A very important question about the behaviour of two lines is taken up for study. Two lines either meet or intercept or not.

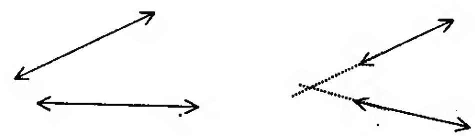


Do they meet? As lines, they meet on the left side. They are in different direction, looking at either way.

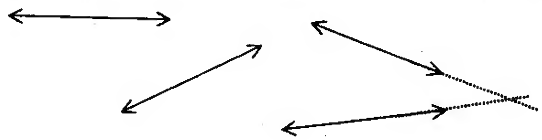
(If line segments are considered, they do not meet.)



Do these lines meet? Yes they do, as could be visualized.



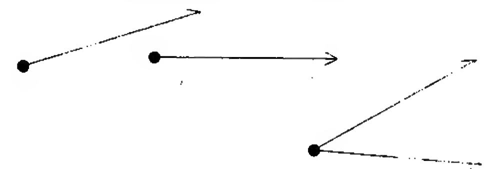
Do these lines meet? Yes they do, as could be visualized.



What about these two lines? They do not meet at all.

Two lines on a plane either meet (or intercept) or do not meet. When they do not meet, they are said to be *parallel*. They are in the same direction looked at either way.

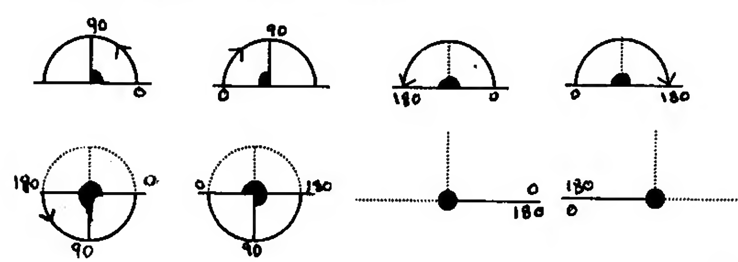
What does one see here? Two rays.



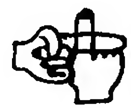
Two rays a common end point. An angle is formed. Its measure is the amount of turning from one ray to another.

Consider turning through quarter round when a right angle is formed. Half a round, when two right angles formed. ¾ th round when three right angles are formed. When full round 4 right angles are formed. It is also called point angle

Note: Giving a right angle in terms of degrees is avoided as division of a right angle into 90^0 is a convention. In metric system it in divided into 100 grades. Only when protractor is used these angles are formed by noting the angle marked in the protractor from either side.



Children can demonstrate these interestingly using four fingers of one left palm with the four fingers of the right palm

(i) 

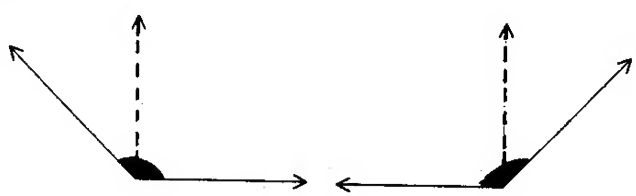
(ii) 

Note: Let children remember what they are asked to do on the playground: right turn, about turns.

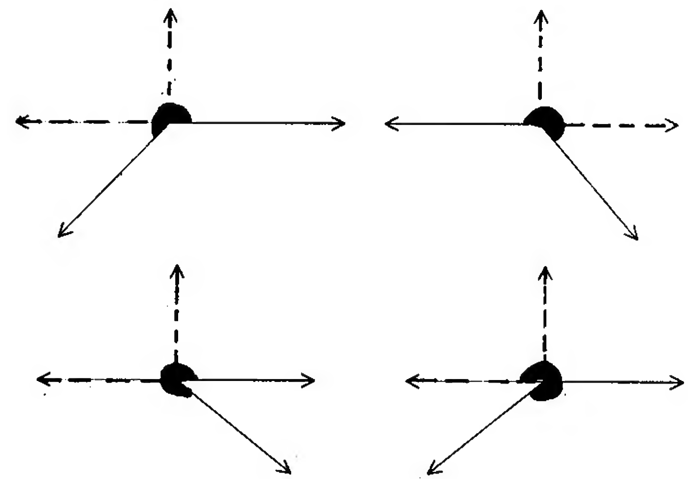
Irregular turning:

When turning is less than a right angle (quarter turn) the angle formed is named acute.

When turning is more than a right angle, the angle formed is obtuse.

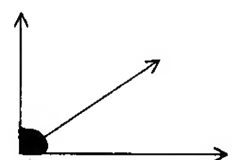


When turning is more than two right angles, the angle formed is named as reflex angle.

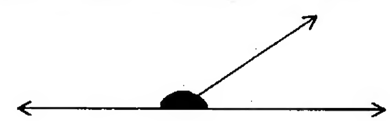


Complementary and supplementary angle:

1. When two angles together make a right angle the two angles are complementary.



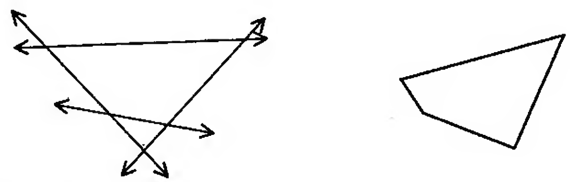
When two angles together make a straight angle, the two angles are supplementary.



FORMATION OF SHAPES.

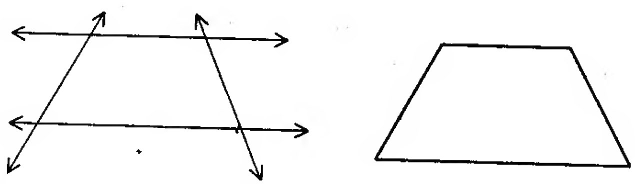
Quadrilateral

1. Take on a plane four lines all in different directions. Note the points of intersection and the figure formed.

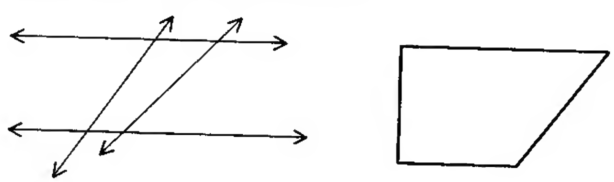


The figure is four sided with four angle-called a quadrilateral.

2. Take on a plane two parallel lines and two more lines not parallel.

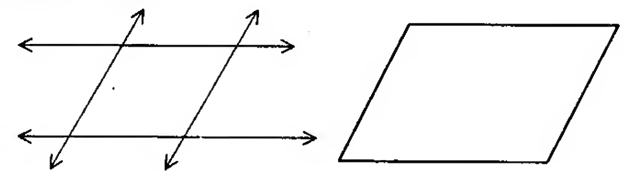


The figure is four sided with one pair of parallel sides.
The figure could have been formed thus also. Here too one pair of sides are parallel.



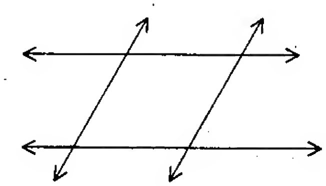
This figure is also called a *trapezium* quadrilateral .

3. Take on a plane two sets of parallel lines, one pair differing in directions from the other pair. Note the figure formed.



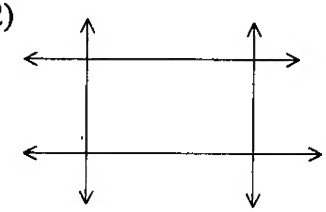
The figure is four sided with two pairs parallel or opposite sides parallel. It is a parallelogram quadrilateral.

Special cases:



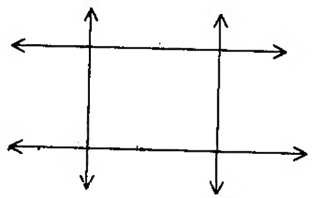
If the figure is so formed that all the sides are of equal length, the figure is a rhombus quadrilateral.

(2)



If the figure is so formed that all the angles are right angles, the figure is a rectangle quadrilateral.

(3)



If the figure is so formed that all the angles are at right angles and all the sides are equal, than the figure is a square quadrilateral.

Note:

(A) Only formed figures could only be drawn but the way that they get formed have to be visualized.

(B) In traditional treatment, line segments are stated to be parallel. Only lines can be parallel and not line segments as considered by the mathematicians.

(C) The figures are all named as quadrilateral just as triangles are named as scalene triangles, isosceles triangle and equilateral triangle. The naming part quadrilateral can be dropped off, once the conceptualization has taken place.

(D) The definitions are yet not precise and they can be made precise in the middle school. Precision requires economy of characteristics requiring description from discovery of properties. For example

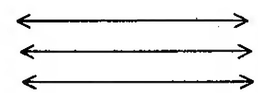
   (i) a rhombus is a quadrilateral with opposite sides parallel and a pair of adjacent sides equal.

   (ii) a rectangle is a quadrilateral with opposite sides parallel and one angle a right angle.

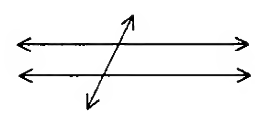
(E) Children, if started on their voyage of geometry this way, will have the confidence and clarity to proceed on their own further study of shapes.

Formation of shapes: triangles:

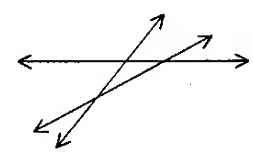
Take three lines. In what ways can they lie with respect to each other. It is an excellent exercise in imagination egged on by intuition.



All parallel

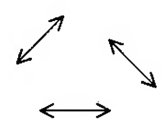


Two of them parallel and one cutting them (called transversal).



No two of them are parallel. They form a triangle.

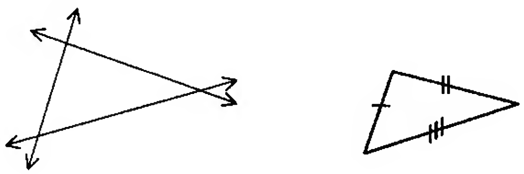
Note:



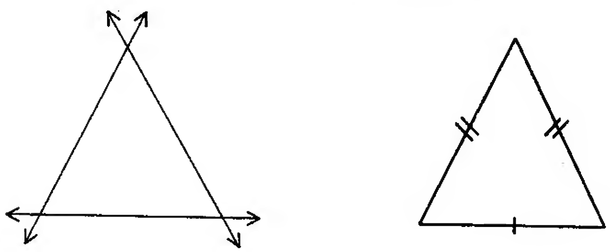
This visual should be taken to show the formation of triangle by three lines all in different directions or no two of which have the same direction.

By placing lines in different directions in different ways and observing the triangles formed in respect (a) sides (b) angles and (c) sides and angles, different kinds can be identified. Use of straight thin sticks (rod or broomsticks) is found helpful.

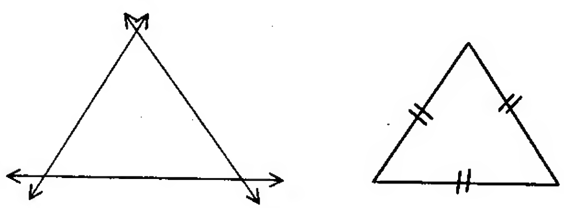
(a) Classification according to sides:



This kind of a triangle has three sides, no two of which are equal and three angles, no two of which have the same measure. This is called a *scalene* triangle.

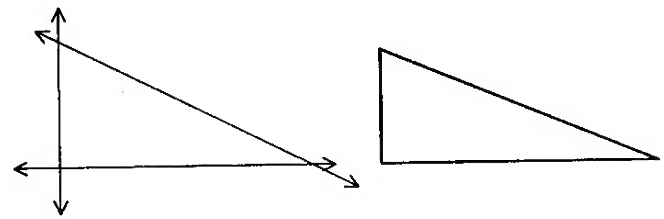


This kind of a triangle has two of its sides equal in length. The fig. Is called an *isosceles* triangle.

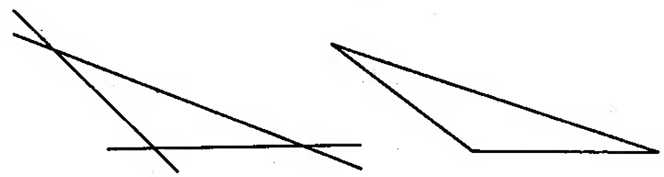


This kind of a triangle has all its three sides of equal length, then the figure is called an *equilateral* triangle.

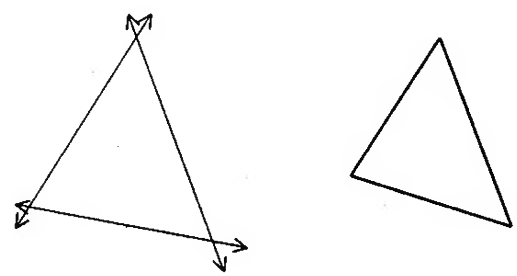
(b) Classification according to angles:



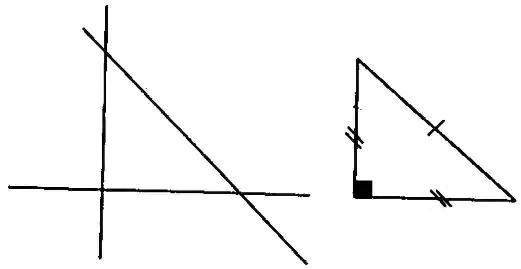
This kind of a triangle has one of its angles a right angle. So the figure is called *right angled triangle.*



This kind of a triangle has one of its angles, an obtuse angle. The figure is called as *obtuse angled triangle.*

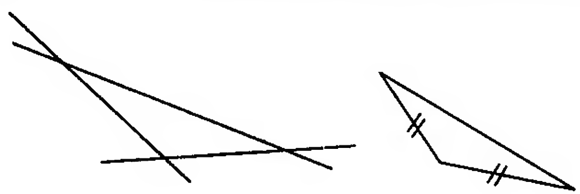


(c) Classification according to sides and angles

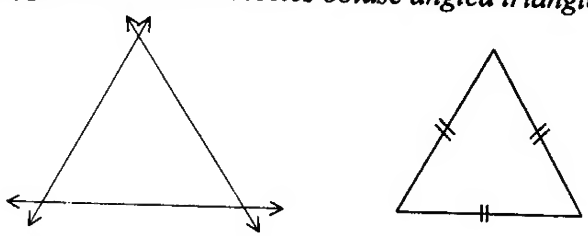


This kind of a triangle has one angle a right angle and two sides forming the right angle are of equal length.

This figure is called as an *Isosceles right angled triangle.*



This kind of a triangle has one angle; an obtuse angle and the two sides forming the obtuse angle are of equal length. This type is called as *isosceles obtuse angled triangle.*



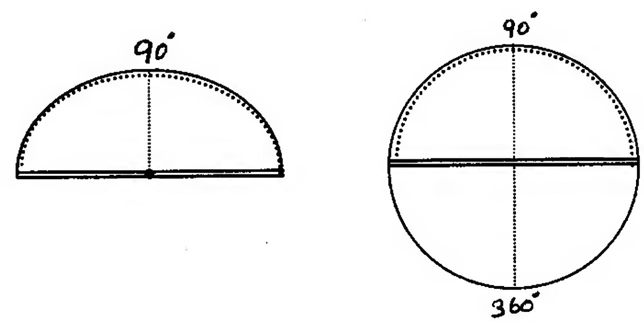
This kind of a triangle has its three angles of equal measure and its three sides of equal length and hence called as *equilateral or equiangular triangle.*

Note: In a formal study later it will be seen that equilateral triangle has to be equiangular and vice-versa.

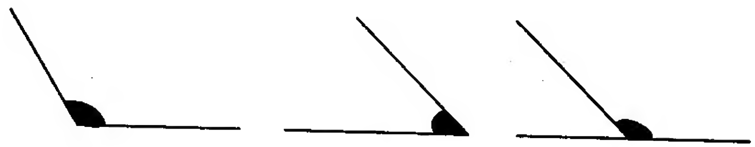
With this back-drop, children have the readiness to take up construction of these figures. It would help if paper folding is done to get these shapes as execution is faster and illuminating, leading to inclusion in practicals test. More of this will be continued in middle school.

Use of protractor should not be left to draw a few angles that may be asked in a test or example.

Protractors are of two shapes, semi circular or circulár. There are two readings at each mark, one giving



obtuse from one end the other acute from the other end. Reading is from either end. Children need to discover the drawing an obtuse angle from a direction can be done drawing its supplementary acute angle from the other direction. To draw this



To draw this obtuse angle

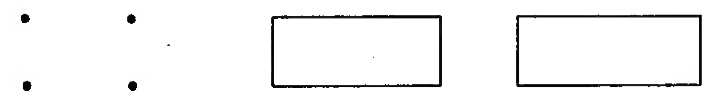
Draw the acute angle supplementary to the obtuse angle

Extend one of the arms to get the required angle.

(13) PERIMETERS AND AREAS

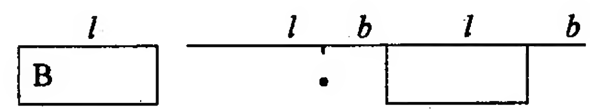
The basic shapes that are to be met with in the development of mathematical concepts centre round the rectangle, the square and the triangle and less so the circle.

In curriculum of Russian schooling, an interesting at the same time valuable distinction is made between shapes of 0 dimension, 1 dimension and 2 dimensions as illustrated below:

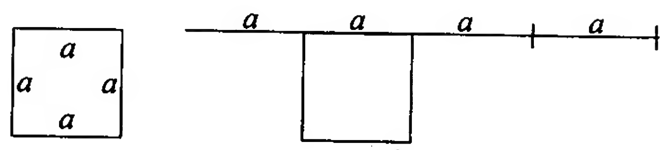


Rectangle of zero dimension, zero perimeter, no area	Rectangle of 1 dimension has perimeter (length and breadth) But no area	Rectangle of 2 dimension has perimeter and area.

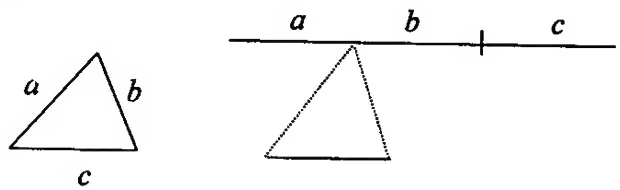
PERIMETER



Perimeter = number of units in length
+ number of units in breadth
+ number of units in length
+ number of units in breadth
= $(l+b+l+b)$ units in short
= $(2l+2b)$ units or
=$2(l+b)$ units of length
(w is also used instead of b sometimes)

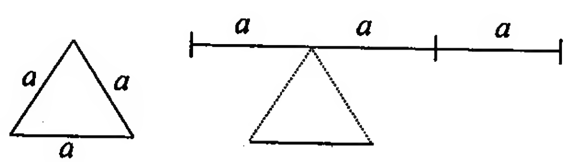


Perimeter of a square is number of units in one side taken 4 times, that is, (a+a+a+a) units or 4a units or 4a units of length.

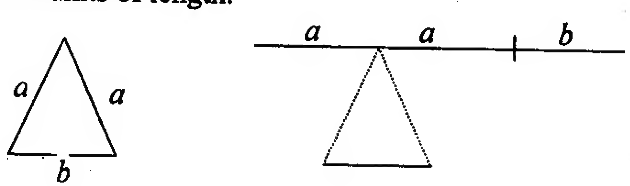


Perimeter of a triangle the number of units of side a + number of units side b + number of units of side c.

= (a+b+c) units of length.



Perimeter of an equilateral triangle is the number of units in one side taken 3 times, that is (a+a+a) units or 3 times side or 3a units of length.



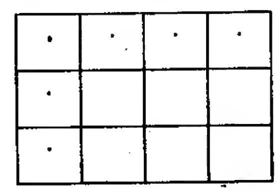
Perimeter of an isosceles triangle is sum of units in (side a+ side a + side b), that is (a+a+b) units or (2a+b) units of length.

AREA:

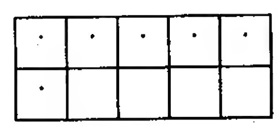
Comparison of two can be done by placing one over the other if possible as in the case of deciding which of two leaves is larger, which of two plates is larger and so on.

To measure the expanse or areas, there is need for units. Since it is easy to cover a plane region by means of unit squares and its parts, square shape is chosen to serve as unit measure of area.

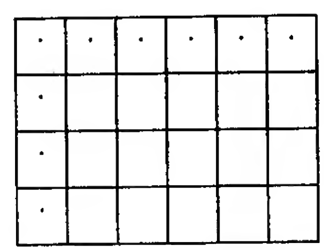
Take check ruled or square ruled networks of rectangular and square shapes.



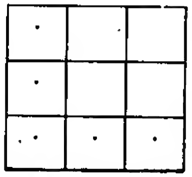
Area	*l*	*b*
12	4	3



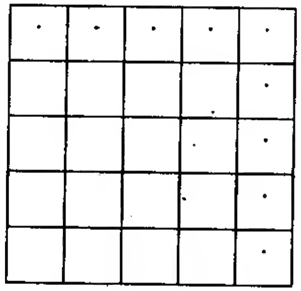
Area	*l*	*b*
10	5	2



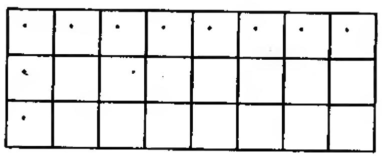
24	6	4



9 3 3



25 5 5



24 8 3

A study of the readings show up thê relation between Area, length and breadth giving the formula for area.

$12 = 4 \times 3$	$25 = 5 \times 5$
$10 = 5 \times 2$	$24 = 8 \times 3$
$24 = 6 \times 4$	$12 = 6 \times 2$
$9 = 3 \times 3$	

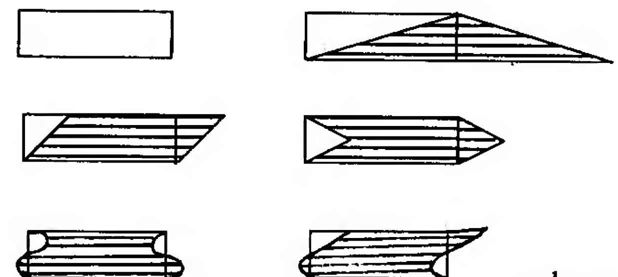
So area of rectangle = number of units in length × number of units in breadth, or l b square units.
(in the case of a square, the formula gets modified to $A = a^2$ square units.)

Note:

1. The coverage of the concept of area is incomplete without exposure to the presence of different shapes having the same area.

   As a matter of fact, primary mathematics learning is educative only when children take up fixing the same area in different shapes. In middle school, the children can take up deeper project.

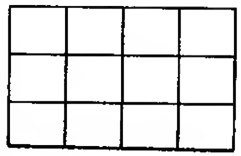
2. Same area in different shapes is not an exercise in construction geometry. But a practical requiring cutting and pasting. Even a competition can be held to recognise students with greatest resourcefulness or creativity. A sample is given below:



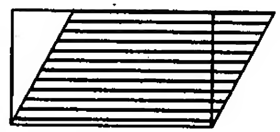
and so on.

Children who have had exposure to halving a rectangle in different ways will find their project of same area in different shapes exciting. Besides shapes of same area having different shapes exciting. Besides shapes of same area having different perimeter, shapes of the same perimeter having different areas and above all shapes of same perimeter and same area being different and challenging.

3. There is need to know the distinction between, say, square centimeters and centimeter squares.
   Observe the illustration given below:



The area is made of cm squares



The area can be given only in square centimeters.

So a figure, built of cm squares has shape with sides in line segments whereas a figure of given square centimeters of area has no particular shape.

4. What about the area of rectangles built of cm squares and their parts? Will the formula of area being number of square units in (length × breadth) applicable?

   This is neglected in syllabi and also by textbook writer. This is simply assumed by fiat.

   Children can make use of unit squares, half unit squares and quarter unit squares and so on.

A few typical cases are presented below:

	2 ½		
2	1	1	½
	1	1	½

Area of the rectangle is equal to the area of (1+1+1+1+½+½) = 5 unit square unit square. Since 2 ½ × 2 = 5, the formula is applicable.

	3 ½			
2 ½	1	1	1	½
	1	1	1	½
	½	½	½	¼

Area of the rectangle is equal to the area of (1 + 1 + 1 + 1 + 1 + 1 + ½ + ½ + ½ + ½ + ½ + ¼) unit squares.
= 8 ¾ sq. units.

Since $2\frac{1}{2} \times 3\frac{1}{2} = \frac{5}{2} \times \frac{7}{2} = \frac{35}{4} = 8\frac{3}{4}$ sq. units. The formula is applicable.

(14) REPEATED MULTIPLICATION – AN OPENING FOR THE PUSHFUL AND BRIGHT STUDENTS

What attracts attention is availability of surprise, suspense and shock. There is a lot of scope for them in mathematical thought. When repeated addition underlies multiplication and repeated subtraction division, the question arises, what about repeated multiplication?

Consider 2

No. of times it is multiplied	out come
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048
12	4096
13	8192
14	16384
15	32768
16	65536
17	131072
18	262144
19	524288
20	1048576

$$32 \times 4 = 128$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \uparrow \\ 5 & + & 2 & = & 7 \end{array}$$

When children discover that two numbers in the outcomes column can be multiplied by adding the corresponding figure in the first column, they get thrilled. This puts them in an enquiring frame of mind to discover division associated with subtraction, raising to the power with multiplication and so on.

Some examples are given below:

$$\begin{array}{ccccc} 256 & \div & 64 & = & 4 \\ \downarrow & & \downarrow & & \uparrow \\ 8 & - & 6 & = & 2 \end{array}$$

$$\begin{array}{ccc} 16^2 & & 256 \\ \downarrow & & \uparrow \\ 4 \times 2 & = & 8 \end{array}$$

They, discover incidentally,

$$\begin{array}{ccccc} 32 & \div & 32 & = & 1 \\ \downarrow & & \downarrow & & \\ 5 & - & 5 & = & 0 \end{array}$$

So 2^0 should mean 1. This can be continued further in high school.

Another significant discovery made is that outcome addition or subtraction of number in the second column can be formed by means of this table.

When children learn that this repeated multiplication table is not to be got by heart but will be given to be used, they have a pleasant surprise.

Some of the problems, they set and try are given below:

$$\frac{1024 \times 128}{2048}$$

$16 \times 64 \times 128$

$32^2 \times 8^3$ and so on.